# A STRICHARTZ INEQUALITY FOR THE SCHRÖDINGER EQUATION ON NON-TRAPPING ASYMPTOTICALLY CONIC MANIFOLDS

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Abstract. We obtain the Strichartz inequality

$$\int_0^1 \int_M |u(t,z)|^4 \, dg(z)dt \leq C\|u(0)\|^4_{H^{1/4}(M)}$$

for any smooth three-dimensional Riemannian manifold $(M,g)$ which is asymptotically conic at infinity and non-trapping, where $u$ is a solution to the Schrödinger equation $iu_t + \frac{1}{2}\Delta_M u = 0$. The exponent $H^{1/4}(M)$ is sharp, by scaling considerations. In particular our result covers asymptotically flat non-trapping manifolds. Our argument is based on the interaction Morawetz inequality introduced by Colliander et al., interpreted here as a positive commutator inequality for the tensor product $U(t,z',z'') := u(t,z')u(t,z'')$ of the solution with itself. We also use smoothing estimates for Schrödinger solutions including one (proved here) with weight $r^{-1}$ at infinity and with the gradient term involving only *one* angular derivative.

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## 1. Introduction

The purpose of this paper is to establish a $L^4$ space-time Strichartz inequality on a class of non-Euclidean spaces, namely smooth three-dimensional asymptotically

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conic Riemannian manifolds $(M, g)$ which obey a non-trapping condition. We make these concepts precise as follows.

**Definition 1.1.** A smooth complete noncompact Riemannian manifold $M$, parameterized[1] by the variable $z := (z^1, \ldots, z^n)$, with metric $g := g_{jk}(z)dz^j dz^k$, is *asymptotically conic* if there exists a fixed compact set $K_0 \subset M$ and an $(n-1)$-dimensional smooth compact Riemannian manifold $(\partial M, h)$ — parameterized by the variable $y := (y^1, \ldots, y^{n-1})$, with metric $h := h_{jk}(y)dy^j dy^k$ — such that the *scattering region* or *asymptotic region* $M \backslash K_0$ can be parameterized as the collar neighbourhood

$$M \backslash K_0 \equiv (0, \epsilon_0) \times \partial M := \{(x, y) : 0 < x < \epsilon_0, y \in \partial M\}$$

for some $\epsilon_0 > 0$, with metric of the form

$$g_{jk}(z)dz^j dz^k = \frac{dx^2}{x^4} + \frac{h_{jk}(x, y)dy^j dy^k}{x^2} \tag{1.1}$$

with the usual summation conventions; here $h_{jk}(x, y)$ is a smooth function on $[0, \epsilon_0) \times \partial M$ such that $h_{jk}(0, y) = h_{jk}(y)$. We will often use the coordinate $r := 1/x$ instead of $x$ in the scattering region.[2]

The metric in co-ordinates $(r, y)$ has the 'asymptotically conic' form

$$g = dr^2 + r^2 h_{jk}(\frac{1}{r}, y)dy^j dy^k.$$

By analogy with Euclidean space, we call $r$ the *radial variable*, and $y$ the *angular variable*; the latter variable is often also denoted $\omega$ or $\theta$ in other texts; here we shall reserve Greek letters for 'cotangent' or 'frequency' co-ordinates and Latin letters for 'spatial' or 'position' co-ordinates. For $V \in T_z M$, we let $|V|_{g(z)}$ denote its length with respect to the metric $g$.

In particular we may compactify $M$ to $\overline{M} := M \cup \partial M$ by identifying $\partial M$ with $\{0\} \times \partial M$ in this co-ordinate chart. If $h_{ij}$ is independent of $x$ (so that $h_{ij}(x, y) = h_{ij}(y)$) we say that $M$ is *perfectly conic near infinity*. Finally, we say that $M$ is *non-trapping* if every geodesic $z(s)$ in $M$ reaches $\partial M$ as $s \to \pm\infty$.

It will be convenient to select a somewhat artificial, but globally defined radial positive weight $\langle z \rangle$, chosen so that $\langle z \rangle$ is equal to $r$ in the scattering region $(0, \epsilon_0) \times \partial M$ and is comparable to $1/\epsilon_0$ in the compact interior region $K_0$, in such a way that the function $\langle z \rangle$ is smooth and obeys the symbol estimates

$$|\nabla_z^j \langle z \rangle| \leq C_j \langle z \rangle^{1-j}$$

for all $j \geq 0$.

*Remark* 1.2. The most important example of an asympotically conic manifold is Euclidean space $(M, g) := (\mathbf{R}^n, \delta)$, with $K_0$ equal to the unit ball $K_0 := \{z : |z| \leq$

[1]We use $z$ as the spatial co-ordinate on $M$, in order to reserve the letter $x$ for the scattering co-ordinate $x := 1/r$. While we phrase some of these definitions in general dimension $n$, we shall focus primarily on the three-dimensional case $n = 3$. We will use the indices $j, k$ to parameterize three-dimensional manifolds $M$ or two-dimensional manifolds $\partial M$ (reserving $i$ for the square root of -1), and the indices $\alpha, \beta$ to parameterize the six-dimensional product manifold $M \times M$ which will play a prominent role in the argument.

[2]Note that $\partial M$ can in fact be recovered from the Riemannian manifold $M$ by identifying points in $\partial M$ with pencils of geodesics that remain at bounded distance apart as $t \to \infty$.

$1\}$, say, with $\partial M$ equal to the unit sphere $S^{n-1} = \{z : |z| = 1\}$ with its standard metric $h$, and scattering co-ordinates

$$x := \frac{1}{|z|}; \quad y := \frac{z}{|z|}; \quad r := |z|$$

for $r > r_0 := 1$. This example is in fact perfectly conic near infinity, and is also clearly non-trapping. More generally, any compact perturbation of Euclidean space will be perfectly conic near infinity, though it may not be non-trapping. Any asympotically Euclidean space $(\mathbf{R}^n, g)$ with decay estimates $|\nabla^j (g - \delta)(z)| \leq C_j \langle r \rangle^{-j-1}$ will also be asymptotically conic, with $\partial M$ equal to the standard sphere $S^{n-1}$; it is quite likely that the smoothness (and decay) assumptions we use can be weakened substantially.

*Remark* 1.3. Note that a non-trapping manifold must in fact be contractible. To see this, we observe that the region $K := \{x \geq \epsilon\}$ is, for $\epsilon$ sufficiently small, a manifold with the same homotopy type as $M$ which is *convex* in the sense required to apply Theorem 4.2 of [25]. This guarantees the existence of a closed geodesic in $M$ provided $\pi_j(K) \neq 0$ for some $j > 0$.

*Remark* 1.4. We note that the class of metrics defined above is precisely the class of *scattering metrics* defined by Melrose [16], with the metric written in a normal form in the scattering region due to Joshi and Sá-Barreto [11].

Let $(M, g)$ be an asymptotically conic manifold, and suppose $u$ is a smooth solution to the (time-dependent) Schrödinger equation

$$iu_t + \frac{1}{2}\Delta_M u = 0 \tag{1.2}$$

on $\mathbf{R} \times M$, where $u(t, z)$ is a complex field, $u(0) \in \dot{C}^\infty(\overline{M})$ is a Schwartz function (i.e. it and its derivatives vanishes to infinite order at $\partial M$) and

$$\Delta_M := \frac{1}{\sqrt{g}} \partial_j \sqrt{g} g^{jk} \partial_k = \mathrm{Tr}_g \nabla^2, \quad g := \det(g_{jk}) \tag{1.3}$$

is the (negative definite) Laplace-Beltrami operator (with $\nabla$ denoting covariant derivatives). It is well known (see e.g. [9]) that if $u(0)$ is Schwartz then there is a unique global Schwartz solution to (1.2), but our estimates will not depend on any Schwartz norms of $u$. We may rewrite (1.2) as

$$u_t = -iHu, \tag{1.4}$$

where $H := -\frac{1}{2}\Delta$ is the Hamiltonian; note that $H$ is positive definite and self-adjoint with respect to the real-valued inner product

$$\langle u, v \rangle_M := \mathrm{Re} \int_M u(z)\overline{v(z)}\, dg(z) \tag{1.5}$$

where $dg(z) := \sqrt{g}\, dz$ is the usual measure induced by the Riemannian metric $g$. In particular we can generate the usual functional calculus of $H$, and define Sobolev spaces $H^s(M)$ for all $s \in \mathbf{R}$ as the class of functions $u$ whose $H^s(M)$ norm $\|u\|_{H^s(M)} := \|(1 + H)^{s/2} u\|_{L^2(M)}$ is finite, where of course we use (1.5) to define $L^2(M)$.

The purpose of this paper is to establish the following local-in-time $L^4$ Strichartz estimate:

**Theorem 1.5.** *For any smooth three-dimensional asymptotically conic non-trapping manifold $M$, and any (Schwartz) solution to* (1.2) *(or* (1.4)*), we have*

$$\int_0^1 \int_M |u(t,z)|^4 \; dg(z)dt \leq C\|u(0)\|^4_{H^{1/4}(M)}. \tag{1.6}$$

Here and in the sequel constants such as $C$ are allowed to depend on the Riemannian manifold $(M,g)$, but not on $u$. In particular once one proves this estimate for Schwartz initial data $u(0)$, one can obtain the same estimate for general $H^{1/4}(M)$ data by the usual limiting argument.

We now contrast Theorem 1.5 with existing results. In Euclidean space $(M,g) = (\mathbf{R}^n, \delta)$ we have the *Strichartz estimate* (see e.g. [14] and the references therein)

$$\|u\|_{L^q_t L^r_x(\mathbf{R}\times\mathbf{R}^n)} \leq C_{n,q,r,s}\|u(0)\|_{H^s(\mathbf{R}^n)} \tag{1.7}$$

whenever

$$\frac{n}{2} - s \leq \frac{2}{q} + \frac{n}{r} \leq \frac{n}{2}; \quad 2 \leq q \leq \infty; \quad 2 \leq r < \infty. \tag{1.8}$$

Here of course

$$\|u\|_{L^q_t L^r_x(\mathbf{R}\times\mathbf{R}^n)} := \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^n} |u(t,z)|^r \; dz\right)^{q/r} dt\right)^{1/q}.$$

In particular the estimate (1.6) is the case $(n,q,r,s) = (3,4,4,1/4)$, but only locally in time: $t \in [0,1]$. In Euclidean space a scaling argument, replacing $u(t,z)$ by $u(\lambda^2 t, \lambda z)$ and letting $\lambda \to \infty$, shows that the estimate (1.6) holds globally in time, that the exponent $H^{1/4}$ on the right-hand side of (1.6) is sharp, and that we can even replace the inhomogeneous Sobolev space $H^{1/4}(\mathbf{R}^3)$ by the homogeneous $\dot{H}^{1/4}(\mathbf{R}^3)$. We believe that the global-in-time estimate also holds in the asymptotically conic nontrapping setting, but we have not attempted to prove it here.

Strichartz estimates are crucial in analyzing the low regularity behavior of nonlinear Schrödinger equations, see for instance [1], [4], and the remarks in Section 8.

In Euclidean space there is also a *local smoothing estimate*

$$\int_{\mathbf{R}} \int_K |\nabla u(t,z)|^2 \; dxdt \leq C_K \|u(0)\|^2_{H^{1/2}(\mathbf{R}^3)} \tag{1.9}$$

for any compact set $K \subset \mathbf{R}^3$; such estimates seem to have first appeared in the work of Constantin and Saut [8], Sjölin [18] and Vega [26]. (Earlier estimates of this 'dispersive smoothing' type, for the KdV equation, date back to work of Kato [13].) In Euclidean space the estimates (1.7), (1.9) are not directly related—indeed they can be proven independently of each other. However, on other manifolds the local smoothing estimate seems to be indispensable for proving optimal Strichartz estimates.

For general manifolds $(M,g)$, the local smoothing estimate is in general false, even if we localize in time, if there are trapped geodesics; this was shown by Doi [10]. However, assuming that $M$ is non-trapping, we have the analogue of (1.9) in arbitrary dimension:

$$\int_0^1 \int_K |\nabla u(t,z)|^2_{g(z)} \; dg(z)dt \leq C_K \|u(0)\|^2_{H^{1/2}(M)}. \tag{1.10}$$

This estimate is due to Craig, Kappeler and Strauss [9] on asymptotically Euclidean space (see also [10]). We shall need this estimate as well as several global versions of it. Some well-known versions (at least for asymptotically flat manifolds; the generalization to asymptotically conic is straightforward) are

$$\int_0^1 \int_M \frac{|\nabla u(t,z)|^2_{g(z)}}{\langle z\rangle^{1+\varepsilon}}\, dg\, dt \le C_\varepsilon \|u(0)\|^2_{H^{1/2}(M)} \tag{1.11}$$

as well as the variant

$$\int_0^1 \int_M \frac{|u(t,z)|^2}{\langle z\rangle^{1+\varepsilon}}\, dg\, dt \le C_\varepsilon \|u(0)\|^2_{H^{-1/2}(M)} \tag{1.12}$$

for $\varepsilon > 0$. These are proved in Appendix II. Furthermore, if $r_0 > 0$ is sufficiently large, we have the Morawetz estimate

$$\int_0^1 \int_{\langle z\rangle > r_0} \frac{|\not\nabla u(t,z)|^2_{g(z)}}{\langle z\rangle}\, dg(z)\, dt \le C \|u(0)\|^2_{H^{1/2}(M)} \tag{1.13}$$

where $\not\nabla := x\nabla_y = \langle z\rangle^{-1}\nabla_y$, which is well-defined in the scattering region, and thus on the domain of integration if $r_0$ is sufficiently large. Note that for the purposes of this estimate, we could equally well define $\not\nabla$ as the orthogonal projection of $\nabla$ to the hyperplane in $T_zM$ orthogonal to $\partial_r$, since $x\nabla_{y_i}$ form a basis for this hyperplane. (For simplicity in writing our estimates, we will henceforth take $\not\nabla$ to be defined globally, and equal to 0 in the compact region $\langle z\rangle < r_0$.)

In this paper, we prove the following generalization of (1.13):

**Lemma 1.6.** *Let $M$ be an asymptotically conic non-trapping manifold. Let $a^{jk}(z)$ be symbols of order zero on $M$, i.e. functions obeying the estimates*

$$|\nabla_z^m a^{jk}(z)| \le C_m \langle z\rangle^{-m}$$

*for all $m \ge 0$. If $u(0) \in H^{1/2}(M)$ and $u$ is the corresponding solution to* (1.2)*, then*

$$\int_0^1 \left| \int_M \frac{a^{jk}(z)\not\nabla_j u(t,z)\overline{\nabla_k u(t,z)}}{r}\, dg(z)\right| dt \le C\|u(0)\|^2_{H^{1/2}(M)}.$$

This improves upon (1.13) since only one of the $u$-derivatives is required to be angular. In flat Euclidean space it was proved recently by Sugimoto [22]. In fact, we prove (and need) a stronger version of this result where the symbols $a^{jk}$ are allowed to depend on an additional parameter; see Lemma 4.3.

The epsilon loss in (1.11), (1.12) is not removable, even in $\mathbb{R}^n$ (although refinements with logarithmic losses are certainly possible); semi-classically, this can be explained by the fact that a particle $z(s)$ moving at unit speed along a geodesic will have a finite integral $\int_{\mathbf{R}} \langle z(s)\rangle^{-1-\varepsilon}\, ds$, whereas the integral $\int_{\mathbf{R}} \langle z(s)\rangle^{-1}\, ds$ diverges logarithmically. The Morawetz estimate (1.13) then corresponds semi-classically to the fact that the slightly smaller integral $\int_{\langle z(s)\rangle > r_0} |v_{\mathrm{ang}}(s)|^2 \langle z(s)\rangle^{-1}\, ds$ avoids the logarithmic divergence and converges absolutely, where $v_{\mathrm{ang}}(s) = r\frac{d}{ds}y(s)$ is the angular component of the velocity. This reflects the fact that the velocity vector of a geodesic becomes purely radial in the asymptotic limit.

Now consider the Strichartz estimate (1.7) on general manifolds. Without the non-trapping condition, one does not have any local smoothing, and one does not expect to obtain the estimate (1.7) with the sharp number of derivatives $s$. However,

it is still possible to obtain a (local-in-time) Strichartz estimate with a loss of derivatives. Indeed, Burq, Gerard and Tzvetkov [3] showed that the estimate

$$\|u\|_{L^q_t L^r_x([0,1]\times M)} \leq C\|u(0)\|_{H^{s+\frac{1}{q}}(M)} \tag{1.14}$$

holds for arbitrary smooth manifolds (trapping or non-trapping) in general dimension. This loss of derivatives, as well as the localization in time, was shown in [3] to be sharp in the case of the sphere (which is in some sense maximally trapping).

The next progress was by Staffilani and Tataru [20], who were able to remove the $1/q$ loss of derivatives for metrics on $\mathbf{R}^n$ which are non-trapping and Euclidean outside a compact set (and also only required $C^2$ regularity of the metric); a key tool was the use of (1.10) to localize to a compact region of space. After the work of Staffilani and Tataru, Burq [2] gave an alternative proof of the same result, and also considered metrics which were asymptotically flat rather than flat outside of a compact set. In this more general setting Burq was almost able to recover the same Strichartz estimate as Staffilani and Tataru, but with an epsilon loss[3] of derivatives:

$$\|u\|_{L^q_t L^r_x([0,1]\times M)} \leq C\|u(0)\|_{H^{s+\varepsilon}(M)}.$$

The idea was to divide $M$ into dyadic shells $R \leq |z| \leq 2R$ and apply a variant of the Staffilani-Tataru argument on each shell, either rescaling $R$ to become 1, or relying on (1.11) instead of (1.10). As remarked earlier, the presence of the $\varepsilon$ in (1.11) is essential, and is related to the epsilon loss of derivatives in Burq's result.

Our result in Theorem 1.5 is slightly stronger than Burq's in the sense that it applies also to asymptotically conic manifolds, and removes the epsilon loss. However, it is also weaker than these previous results because it is restricted to the specific exponents $(n, q, r, s) = (3, 4, 4, 1/4)$ and cannot obtain the full range (1.8) of the estimate (1.14), although one can obtain a subset of this range by applying Sobolev embedding to (1.6) and also using the energy identity $\|u(t)\|_{L^\infty_t L^2(\mathbf{R}\times M)} = \|u(0)\|_{L^2(M)}$. We conjecture however that the full set of Strichartz estimates can be recovered with no loss of derivatives for non-trapping asymptotically conic (or asymptotically flat) manifolds in arbitrary dimension.

Unlike the previous results, the proof of Theorem 1.5 does not proceed via construction of a parametrix or proving any dispersive estimates on the fundamental solution. Instead, we use the interaction Morawetz inequality approach introduced in [6], [7] in the context of a non-linear Schrödinger equation in Euclidean space. We re-interpret this approach in the language of positive commutators applied to the tensor product $U(t, z', z'') := u(t, z')u(t, z'')$ of the solution $u$ with itself, and then modify it to asymptotically conic manifolds.

There will be a number of technical difficulties in carrying out the above strategy, first in understanding the error terms generated by the non-zero curvature, and second the problem of avoiding the singularities of the metric function $d_M(z', z'')$ once this distance exceeds the radius of injectivity. Unlike the argument in [2], our approach cannot afford to localize the radial variable to dyadic blocks $R < r < 2R$ as this would create error terms with radial derivatives in them, which cannot be controlled by (1.11) without losing an epsilon too many derivatives. In fact we can only localize $r$ once, to the region $r > r_0$, though we can localize the *angular*

[3]The endpoint $q = 2$ is also not obtained, for a rather technical reason involving the causality cutoff $s < t$ in Duhamel's formula.

variable $y$ to a small set [4]. The understanding of the geometry in this regime — in particular, controlling the derivatives of the distance function $d_M((r', y'), (r'', y''))$ when $y', y''$ are close, and $r', r''$ are both large but not necessarily comparable — appears to be absolutely essential for obtaining bounds such as (1.7) with no loss of derivatives whatsoever. Thus our argument requires a certain amount of geometric machinery which we have placed in an appendix (Section 9).

It is well known that Strichartz estimates can be used in the theory of nonlinear Schrödinger equations, and we briefly discuss a modest application of this estimate to a class of non-linear problems in Section 8. However as our Strichartz estimate is restricted to $L^4$ in space and time, the application is admittedly somewhat limited. It is still an important problem as to whether one can obtain the full range of Strichartz estimates (with no derivative loss) on general non-trapping manifolds, and whether they can be extended to be global in time; such a result will have substantially more applications to non-linear Schrödinger equations.

## 2. Positive commutators and Morawetz identities

In this section we review the basic method of positive commutators in the context of the time-dependent Schrödinger equation, and also indicate how it is connected to estimates of Morawetz type; we will rely on these methods, together with detailed analysis of the various terms arising from such methods, to prove Theorem 1.5. In this discussion $M$ is a general smooth Riemannian manifold with compactification $\overline{M}$. In our applications $M$ will either be a three-dimensional non-trapping asymptotically conic manifold, or else the product manifold $M \times M$ which is six-dimensional and non-trapping, but not asymptotically conic (as we shall see, it has a 'corner' at infinity which we will have to blow up to analyze). To emphasize that we may be working in the product setting, we will use Greek indices $\alpha, \beta, \gamma$ to parameterize the manifold rather than Latin indices $j, k, l$.

Let $u$ be a solution to the Schrödinger equation (1.2) (or (1.4)); to avoid technical issues let us assume that $u \in \dot{C}^\infty(\overline{M})$, that is, $u$ is rapidly decreasing together with all derivatives as $z \to \infty$. Given a pseudo-differential[5] operator $A$ on $M$, we can easily verify[6] the *Heisenberg equation*

$$\partial_t \langle Au(t), u(t) \rangle_M = \langle i[H, A]u(t), u(t) \rangle_M,$$

where $[H, A] := HA - AH$ is the usual commutator. Integrating this we obtain the inequality

$$\int_0^1 \langle i[H, A]u(t), u(t) \rangle_M \; dt \leq 2 \sup_{t \in [0,1]} |\langle Au(t), u(t) \rangle_M|. \tag{2.1}$$

[4]Even doing this localization creates some unpleasant error terms in which only *one* of the derivatives is angular. We shall control such terms using Lemma 1.6, or more precisely Lemma 4.3, which only requires one angular derivative.

[5]In fact, we shall mostly be able to rely on *classical* differential operators $A$ for our commutants; pseudo-differential commutants will only be needed to prove the local smoothing estimates necessary to control errors arising from the classical commutant.

[6]Note the presence of the real part in (1.5) allows us to obtain this even when $A$ is not self-adjoint. In practice, we will only deal with operators $A$ whose *principal* symbols are real, so that they are self-adjoint modulo lower order terms.

The strategy of the method of positive commutators is to select a good commutant $A$, chosen so that the commutator $i[H, A]$ is positive (semi-)definite, possibly modulo a lower order error term, and then apply (2.1) to obtain a non-trivial spacetime bound on $u$. From the pseudo-differential calculus and the sharp Gårding inequality, one only expects to be able to do this if the symbol $\sigma(A)$ of $A$ increases along the bicharacteristic flow of $H$ (i.e. geodesic flow in phase space). At first glance this method only seems able to obtain quadratic estimates on $u$ (e.g. the method can be used to deduce (1.10) – (1.13)), but we will also be able to obtain quartic estimates (and in particular the $L^4$ bound (1.6)) by the trick of replacing $u$ by the tensor product $U(t, z', z'') := u(t, z')u(t, z'')$, and working in the six-dimensional product manifold $M \times M$.

For future reference we compute some commutators of $H$ with multiplier operators.

**Lemma 2.1.** *Let $a(x)$ be a real-valued tempered distribution on $M$, thought of as a multiplier operator $(af)(x) := a(x)f(x)$ on Schwartz functions. Then we have the single commutator identity*

$$i[H, a] = i\langle(\nabla a), \nabla\rangle_g + i(Ha) = i(\nabla^\alpha a)\nabla_\alpha + i(Ha) \tag{2.2}$$

*and the double commutator identity*

$$-[H, [H, a]] = -\nabla_\beta \operatorname{Hess}(a)^{\alpha\beta}\nabla_\alpha - (H^2 a) \tag{2.3}$$

*where* $\operatorname{Hess}(a)^{\alpha\beta}$ *is the symmetric tensor*

$$\operatorname{Hess}(a)^{\alpha\beta} = (\nabla da)^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}\Big(\partial_\gamma\partial_\delta a + \Gamma^\rho_{\gamma\delta}\partial_\rho a\Big). \tag{2.4}$$

*Proof.* The proof of (2.2) is elementary. To prove (2.3), observe that both sides of (2.3) are second-order self-adjoint operators (with respect to $\langle,\rangle_M$) with real coefficients. Also, by applying both sides to the constant function 1 we see that the constant terms of (2.3) match. Thus it will suffice to show that the principal symbols of both sides match, since this would mean that the difference of the left-hand side and right-hand side is a self-adjoint first-order operator with real coefficients and no constant term, which is necessarily zero.

To compute the principal symbols at a point $p$, we pass to Riemannian normal coordinates about $p$. Then up to second order at $p$, $g_{\alpha\beta} = \delta_{\alpha\beta}$, and inserting (2.2), we see that the LHS of (2.3) becomes

$$-(\partial_\alpha\partial_\beta(a))\partial_\alpha\partial_\beta,$$

modulo lower order terms. That the symbol agrees with minus the Hessian follows, as Christoffel symbols in (2.4) vanish at $p$.

□

*Remark* 2.2. Because $i[H, a]$ is self-adjoint, we know that $\int \langle i[H, a]u, u\rangle\, dg(z)$ is real for any test function $u$. Since $\int \langle i(Ha)u, u\rangle\, dg(z)$ is clearly imaginary, we thus see from (2.2) that

$$\langle i[H, a]u, u\rangle_M = \operatorname{Im} \int_M \langle\nabla a, \nabla u\rangle_g \overline{u}\ dg. \tag{2.5}$$

More generally, if $b$ is another real function, then

$$\langle bi[H, a]u, u\rangle_M = \operatorname{Im} \int_M b\langle\nabla a, \nabla u\rangle_g \overline{u}\ dg. \tag{2.6}$$

*Remark* 2.3. The Hessian $\mathrm{Hess}(a)^{\alpha\beta}$ measures the convexity of the function $a(z)$ with respect to the geodesic flow. Indeed, if $z(s)$ is the curve of a unit-speed geodesic in $M$, and $v^\alpha(s) := \frac{d}{ds} z^\alpha(s)$ is the velocity vector, then one has

$$\frac{d^2}{ds^2} a(z(s)) = \mathrm{Hess}(a)_{\alpha\beta}(z(s)) v^\alpha(s) v^\beta(s). \tag{2.7}$$

This can be seen either from (2.4) and the geodesic flow equation, or alternatively from (2.3) and the symbol calculus.

*Example* 2.4 (Euclidean one-particle Morawetz inequality; [17], [15]). Consider flat three-space $M := \mathbf{R}^3$. Let $a(z) := |z|$, the distance function to the origin. A simple computation shows that

$$\mathrm{Hess}(a)^{jk}(z) = \partial_j \partial_k a(z) = \frac{1}{|z|} - \frac{z^j z^k}{|z|^3}, \text{ so that}$$

$$\mathrm{Hess}(a)^{jk}(z) \overline{\partial^j u(t,z)} \partial^k u(t,z) = \frac{|\nabla\!\!\!/ u(t,z)|^2}{|z|}$$

where $\nabla\!\!\!/$ denotes the angular gradient as before. Also,

$$-H^2 a = -\frac{1}{4} \Delta\Delta a = 2\pi\delta,$$

where $\delta$ is the Dirac mass at the origin in $\mathbf{R}^3$. Thus by (2.3), (2.5) we have

$$\langle -[H,[H,a]]u, u\rangle_M = 2\pi |u(t,0)|^2 + \int_{\mathbf{R}^3} \frac{|\nabla\!\!\!/ u(t,z)|^2}{|z|}\ dz$$

and

$$\langle i[H,a]u,u\rangle_M = \mathrm{Im} \int \overline{u}(t,z) \frac{z}{|z|} \cdot \nabla u(t,z)\ dz.$$

Since $\nabla$ maps $\dot H^{\frac{1}{2}}(\mathbf{R}^3)$ to $\dot H^{-\frac{1}{2}}(\mathbf{R}^3)$, and $\frac{z}{|z|}$ is a bounded multiplier[7] on $\dot H^{-\frac{1}{2}}(\mathbf{R}^3)$ we have

$$\left| \int_{\mathbf{R}^3} \overline{u}(t,z) \frac{z}{|z|} \cdot \nabla u(t,z)\ dz \right| \leq C \|u(t)\|^2_{\dot H^{1/2}(\mathbf{R}^3)};$$

since the $\dot H^{1/2}(\mathbf{R}^3)$ norm is conserved, setting $A = i[H,a]$ in (2.1) we thus obtain the *Morawetz inequality*

$$\int_0^1 |u(t,0)|^2\ dt + \int_0^1 \int_{\mathbf{R}^3} \frac{|\nabla\!\!\!/ u(t,z)|^2}{|z|}\ dzdt \leq C \|u(0)\|^2_{\dot H^{1/2}(\mathbf{R}^3)}.$$

In fact, in Euclidean space this particular estimate extends from the time interval $[0,1]$ to all of $\mathbf{R}$ by the same argument. Note in particular that we have proven (1.13) for Euclidean spaces. A similar argument can be used to derive (1.13) for general non-trapping manifolds, as well as (1.11), if one already knows the dispersive smoothing estimate (1.10); however the estimate (1.10) cannot in general be proven without introducing commutants which are pseudo-differential operators rather than classical differential operators.

[7] This can be seen by first verifying boundedness on $L^2(\mathbf{R}^3)$ and $\dot H^1(\mathbf{R}^3)$ (using Hardy's inequality) and then using interpolation and duality.

*Example* 2.5 (Euclidean two-particle (interaction) Morawetz inequality; [6], [7]). We now consider a variant of the Morawetz inequality. We let $M$ be the six-dimensional space $M := \mathbf{R}^3 \times \mathbf{R}^3$, which we parameterize as $z = (z', z'')$ where $z', z'' \in \mathbf{R}^3$ and $z \in \mathbf{R}^3 \times \mathbf{R}^3$; observe that the Hamiltonian $H := -\frac{1}{2}\Delta_{\mathbf{R}^3 \times \mathbf{R}^3}$ on this manifold splits as $H = H_{z'} + H_{z''}$ where $H_{z'} := -\frac{1}{2}\Delta_{z'}$ and $H_{z''} := -\frac{1}{2}\Delta_{z''}$.

Let $a : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ be the Euclidean distance function $a(z', z'') = |z' - z''|$. The function $a(z', z'')$ is clearly (weakly) convex under geodesic flow of $z'$ and $z''$ (as can be seen for instance using center of mass co-ordinates), thus the tensor $\mathrm{Hess}(a)^{\alpha\beta}$ is positive semi-definite. In particular we have

$$\mathrm{Hess}(a)^{\alpha\beta}(z)\overline{\partial^\alpha U(t, z', z'')}\partial^\beta U(t, z', z'') \geq 0. \tag{2.8}$$

One can also compute

$$-H^2 a = 8\pi\delta(z' - z''), \tag{2.9}$$

thus $-H^2 a$ is a Dirac mass on the diagonal $\{(z', z'') \in \mathbf{R}^3 : z' = z''\}$ of $\mathbf{R}^3 \times \mathbf{R}^3$. The same reasoning as in the previous example yields the product Morawetz inequality

$$\int_0^1 \int_{\mathbf{R}^3} |U(t, z, z)|^2 \; dzdt \leq C\|U(0)\|^2_{\dot H^{1/2}(\mathbf{R}^3 \times \mathbf{R}^3)}.$$

This inequality holds for all solutions $U(t, z', z'')$ to the Schrödinger equation on $\mathbf{R}^3 \times \mathbf{R}^3$. If we now specialize to the tensor product $U(t, z', z'') := u(t, z')u(t, z'')$ of an $\mathbf{R}^3$ solution $u(t, z)$ of the Schrödinger equation[8], we obtain

$$\int_0^1 \int_{\mathbf{R}^3} |u(t, z)|^4 \; dzdt \leq C\|u(0)\|^2_{L^2(\mathbf{R}^3)}\|u(0)\|^2_{\dot H^{1/2}(\mathbf{R}^3)}; \tag{2.10}$$

after a Littlewood-Paley decomposition (see Proposition 3.1 below), this will imply in particular the Strichartz estimate (1.6). Again in the Euclidean space setting it is possible to extend the time interval from $[0, 1]$ to all of $\mathbf{R}$.

We have just proven Theorem 1.5, for Euclidean space $\mathbf{R}^3$. It is thus tempting to adapt the above proof to more general manifolds. The most obvious thing to try is to repeat the above argument, with the manifold $\mathbf{R}^3 \times \mathbf{R}^3$ replaced of course by $M \times M$ (with the product metric $g \oplus g$), and with the function $|z' - z''|$ replaced by the distance function $d_M(z', z'')$. This works well locally, when $z'$ and $z''$ are close together; the main issue is to verify that $d_M(z', z'')$ remains essentially geodesically convex, and that the error terms can be treated by the one-particle estimates (1.11) – (1.13). Difficulties arise however in the scattering region, requiring more careful control on the geometry of the distance function on asymptotically conic manifolds over long distances, as well as requiring a refinement of Lemma 1.6, but we shall address these issues later.

In Euclidean space one can also develop $k$-particle versions of the Morawetz inequality for any $k \geq 2$ by working in center-of-mass coordinates, with the function $a$ being the standard deviation of the $k$ positions $z_1, \ldots, z_k$. This leads to a useful

[8]Note here we are using a very special property of the Schrödinger with itself, namely that tensor products of solutions to the Schrödinger equation remain solutions to a Schrödinger equation (but now on the product manifold of twice the dimension). This property is not shared by other equations such as the wave or Klein-Gordon equation, and it remains a challenge to extend this type of argument to those equations.

monotonicity formula in $d$ dimensions when $d(k-1) \geq 3$ (for instance the 4-particle Morawetz inequality gives an $L^8_{t,z}$ estimate in one dimension). In this paper, however, we will confine ourselves to the two-particle problem.

## 3. Littlewood-Paley reduction

In the previous section we discussed the method of positive commutators, and showed how it can be used to prove $L^4_{t,x}$ type estimates such as (1.6). It is then tempting to apply the same argument immediately to general manifolds, using the distance function $d_M(z', z'')$ as a substitute for $|z' - z''|$ as mentioned above. Before we do so, however, it is convenient to make a Littlewood-Paley reduction which allows us to restrict the solution $u$ to a single dyadic frequency range $[2^{2k-2}, 2^{2k+2}]$ in the spectrum of $H$. This reduction is standard (see e.g. [19], Lemma 5.1 and the ensuing discussion).

**Proposition 3.1.** *Suppose we have*

$$\int_0^1 \int_M |u(z,t)|^4 \, dg(z) \, dt \leq C \, 2^k \, \|u(0)\|^4_{L^2(M)}, \tag{3.1}$$

*for all solutions $u(z,t)$ of (1.4) with $u(z,0)$ in the range of the spectral projection $\chi_{[2^{2k-2}, 2^{2k+2}]}(H)$, and all $k \geq 1$. Then Theorem 1.5 holds.*

*Proof.* The solution to (1.4) is given by $u(t) = e^{-itH}u(0)$. Furthermore we have a Littlewood-Paley decomposition

$$\mathrm{Id} = \phi_0(H) + \sum_{k=1}^{\infty} \psi_k(H)$$

where $\phi_0$ is a compactly supported bump function, and $\psi_k$ is a bump function adapted to the interval $[2^{2k-2}, 2^{2k+2}]$. This induces a decomposition

$$u(0) = u(0)^{(0)} + \sum_{k=1}^{\infty} u(0)^{(k)}, \quad u(0)^{(0)} = \phi_0(H)u(0), \quad u(0)^{(k)} = \psi_k(H)u(0).$$

Informally, $\psi_k(H)$ is the spectral projection to functions of frequency comparable to $2^k$.

The contribution of the low-frequency term $u(0)^{(0)}$ can be handled very easily using the Sobolev embedding $\|u\|_{L^4} \leq C\|\nabla u\|_{L^2}$ in four dimensions:

$$\|e^{-itH}u(0)^{(0)}\|_{L^4_{t,x}([0,1]\times M)} \leq C\|\nabla_{z,t}\big(e^{-itH}\phi_0(H)u(0)\big)\|_{L^2([0,1]\times M)}$$
$$\leq C\|(1+H)e^{-itH}\phi_0(H)u(0)\|_{L^2([0,1]\times M)} \leq C\|u(0)\|_{L^2(M)} \leq \|u(0)\|_{H^{1/4}(M)}.$$

So consider the high-frequency terms

$$\|\sum_{k=1}^{\infty} e^{-itH}u(0)^{(k)}\|_{L^4_{t,x}([0,1]\times M)}. \tag{3.2}$$

We use the Littlewood-Paley estimate

$$\|\sum_k \psi_k(H) f\|_{L^4(M)} \leq C\|(\sum_k |\psi_k(H)f|^2)^{1/2}\|_{L^4(M)}$$

(see e.g. [3] or [21], chapter VI, section 7.14; the point is that the heat kernels $e^{-tH}$ or resolvents $(H-\lambda)^{-1}$ enjoy good decay and regularity properties, and thus the

Littlewood-Paley square function operator is of Calderón-Zygmund type), to bound (3.2) by

$$C\|(\sum_{k=1}^{\infty}|e^{-itH}u(0)^{(k)}|^2)^{1/2}\|_{L^4_{t,x}([0,1]\times M)};$$

we can move the $l^2$ sum outside of the $L^4$ norm to bound this by

$$C\Big(\sum_{k=1}^{\infty}\|e^{-itH}u(0)^{(k)}\|^2_{L^4_{t,x}([0,1]\times M)}\Big)^{1/2}.$$

The estimate (3.1) implies that (3.2) is bounded by

$$C\Big(\sum_{k=1}^{\infty}2^{k/2}\|u(0)^{(k)}\|^2_{L^2(M)}\Big)^{1/2}.$$

From the spectral localization of $u(0)^{(k)}$, we can estimate this by

$$C\Big(\sum_{k=1}^{\infty}\|(1+H)^{1/8}u(0)^{(k)}\|^2_{L^2(M)}\Big)^{1/2}.$$

Since $u(0)^{(k)} = \psi_k(H)u$, we see from spectral calculus that this expression is $O(\|(1+H)^{1/8}u(0)\|_{L^2(M)}) = O(\|u(0)\|_{H^{1/4}})$, as desired. $\square$

Henceforth we fix $k > 0$ and write $u = u^{(k)}$, thus $u$ is implicitly localized to the portion of the spectrum of $H$ lying in the interval $[2^{2k-2}, 2^{2k+2}]$. Our task is now to prove (3.1). Observe from this spectral localization (and the unitarity of the operators $e^{-itH}$) that we have the bounds

$$c2^{ks}\|u(0)\|_{L^2(M)} \leq \|u(t)\|_{H^s(M)} \leq C_s 2^{ks}\|u(0)\|_{L^2(M)} \tag{3.3}$$

for all $s \in \mathbf{R}$ and $t \in [0,1]$. In particular, we have

$$\|\nabla u(t)\|_{L^2(M)} \leq C\|u(t)\|_{H^1(M)} \leq C2^k\|u(0)\|_{L^2(M)} \tag{3.4}$$

(the first inequality being easily verified by an integration by parts). These inequalities give us a significant amount of freedom to estimate various factors of $u$ in various Sobolev spaces, which will be very convenient in the argument that follows. Indeed, because of these estimates, the *location* of the derivatives in our expressions to be estimated will no longer be particularly relevant; what matters[9] is the *number* of such derivatives (which range between 0 and 2, with the terms with two derivatives of course being the most difficult) as well as their *type* (angular derivatives will, in general, be easier to handle than radial derivatives, thanks for instance to (1.13)).

## 4. Overview of proof of Theorem 1.5

Now that we have localized $u$ to a single frequency range, we apply the method of positive commutators discussed in Section 2. We will in fact use this method twice; once for the near region $K_0$, and once the scattering region $M\backslash K_0$, using different commutants for the two regions.

[9]Of course, weights such as $1/\langle z\rangle$ and $1/\langle z'\rangle$ will also be important to keep track of.

Let $0 < \varepsilon \ll 1$ be a small constant to be chosen later. We use $Z$ to denote the quantity

$$Z := \varepsilon^{-2} 2^k \|u(0)\|_{L^2(M)}^4 + \varepsilon \int_0^1 \int_M |u(z,t)|^4 \, dg(z) \, dt. \tag{4.1}$$

Clearly, to prove (3.1), it suffices to show that

$$\int_0^1 \int_M |u(z,t)|^4 \, dg(z) \, dt = O(Z), \tag{4.2}$$

since for $\varepsilon$ small enough we can absorb the second term in (4.1) into the left-hand side.

It remains to prove (4.2). We first exploit the compactness of $\overline{M}$ to work on smaller regions.

**Proposition 4.1** (Local $L^4$ bound)**.** *For every $z_0 \in M$ we have*

$$\int_0^1 \int_{B(z_0,\eta)} |u(z,t)|^4 \, dg(z) \, dt \leq C_{z_0,\eta} Z$$

*if $\eta$ is sufficiently small (depending on $z_0$), where $B(z_0,\eta)$ is the ball of radius $\eta$ centered at $z_0$.*

**Proposition 4.2** (Scattering $L^4$ bound)**.** *We have*

$$\int_0^1 \int_{\langle z \rangle > r_0} |u(z,t)|^4 \, dg(z) \, dt \leq C_{r_0} Z$$

*if $r_0$ is sufficiently large.*

From these two propositions and the compactness of $\overline{M}$ we easily obtain (4.2) as desired.

It remains to prove the propositions. Proposition 4.1 is the easier of the two and is proven in Section 5, by applying the positive commutator method on the product manifold $M \times M$ (with Hamiltonian $H_{M\times M} = H_{z'} + H_{z''}$) to the product solution $U(t,z',z'') := u(t,z')u(t,z'')$, and with the commutant $A_{\text{near}} = A_{\text{near},z_0,\eta}$ defined by

$$A_{\text{near}} := \varphi(z',z'') i[H_{M\times M}, d_M(z',z'')], \tag{4.3}$$

where $\varphi(z',z'')$ is a smooth non-negative cutoff to the region $z',z'' \in B(z_0,\eta)$ which equals 1 on the region $z',z'' \in B(z_0,\eta/2)$. Note that while the first-order operator $A_{\text{near}}$ is not quite self-adjoint, it does have real principal symbol and its commutator will turn out to be positive definite (modulo lower order terms). The commutator $[H_{M\times M}, A_{\text{near}}]$ turns out in fact to be quite tractable, as most of the error terms generated can be treated by the standard local smoothing estimates (1.11), (1.12).

Proposition 4.2 is proven in a similar manner but involves more technical difficulties (in particular, one needs much more precise control on the derivatives of the metric function $d_M$ near infinity). The proof applies the positive commutator method on $M \times M$ to the same product function $U$ discussed before, but with the commutant $A_{\text{asympt}} = A_{\text{asympt},r_0}$ now given by

$$A_{\text{asympt}} = \chi(z',z'')\psi(y',y'') i[H_{M\times M}, d_M(z',z'')] \tag{4.4}$$

where $\chi(z',z'')$ is a smooth non-negative cutoff to the product scattering region $\{\langle z' \rangle, \langle z'' \rangle > r_0/2\}$ which equals 1 on the slightly smaller region $\{\langle z' \rangle, \langle z'' \rangle \geq r_0\}$,

and $\psi$ is a smooth non-negative cutoff[10] to the region $\{d_{\partial M}(y', y'') \leq 2\eta\}$ which equals one on $\{d_{\partial M}(y', y'') \leq \eta\}$, where $\eta > 0$ is a small number (less than a quarter the injectivity radius of $\partial M$) to be chosen later. Note that while $\chi\psi$ localizes $y'$ and $y''$ to be close to each other, it does not localized $r'$ and $r''$ to be similarly close (although we have made them both large); it turns out that we cannot afford to localize further as this will generate error terms that are too difficult to estimate.

In the asymptotic regime, when controlling $[H, A_{\mathrm{asympt}}]$, terms arising from derivatives hitting cutoff functions are harder to control, mainly because there are many terms which decay only like $1/\langle z'\rangle$ or $1/\langle z''\rangle$, and the estimates (1.11), (1.12) instead require decay of $1/\langle z'\rangle^{1+\varepsilon}$, $1/\langle z''\rangle^{1+\varepsilon}$. One can use (1.13) to control those terms which have two angular derivatives. Unfortunately there are also mixed terms involving exactly *one* angular derivative; these terms come from one derivative hitting the angular cutoff $\psi$ and one hitting the distance function $d_M$. A typical such term has the form

$$\int_0^1 \int_{r', r'' \geq r_0/2} \frac{a^{ij}(z', z'') \not\nabla_i u(t, z') \overline{\nabla_j u(t, z')}}{r'} |u(t, z'')|^2 \, dg' \, dg'' \, dt,$$

where $a^{ij}(z', z'')$ are symbols of order 0 in $z'$, uniformly in $z''$ (see Proposition 9.4 below), and $\not\nabla_i$ represents a component of the angular derivative $\frac{1}{r}\nabla_y$. If both derivatives were angular then such a term could be controlled by the single particle Morawetz estimate (1.13), however this estimate is insufficient when one of the derivatives has a radial component. Fortunately these mixed terms can still be handled by the following new endpoint smoothing estimate:

**Lemma 4.3.** *Let $W$ be an arbitrary index set (possibly infinite). For each $w \in W$, let $a_w^{jk}(z)$ be a tensor-valued symbol ($j, k$ range over the co-ordinate indices of $\partial M$ and $M$ respectively) supported on the scattering region $\{\langle z\rangle > r_0\}$ for sufficiently large $r_0$ which are uniformly symbols of order 0 in the sense that*

$$|\nabla_z^m a_w^{jk}(z)| \leq C_m \langle z\rangle^{-m}$$

*for all[11] $m \geq 0$, where the constants $C_m$ do not depend on $w$. Then we have*

$$\int_0^1 \sup_{w \in W} \left| \int_M \frac{a_w^{jk}(z) \not\nabla_j u(t, z) \overline{\nabla_k u(t, z)}}{r} \, dg(z) \right| dt \leq C \|u(0)\|^2_{H^{1/2}(M)}. \tag{4.5}$$

*We also have the variant*

$$\int_0^1 \sup_{w \in W} \left| \int_M \frac{a_w^{j}(z) \not\nabla_j u(t, z) \overline{u(t, z)}}{r} \, dg(z) \right| dt \leq C \|u(0)\|^2_{L^2(M)} \tag{4.6}$$

*where $a_w^j$ obeys similar bounds but without the additional index $k$.*

This lemma appears to be new even when $W$ is a singleton set, and may be of independent interest. Morally speaking, it corresponds semi-classically to the fact that $\int_{\langle z\rangle > r_0} |v_{\mathrm{ang}}(s)|/\langle z(s)\rangle \, ds$ converges for all geodesics $z(s)$ of unit speed (cf. the discussion after Lemma 1.6); however the result is still rather subtle, since the inequalities fail if the absolute value signs are brought inside the $z$-integral; see Proposition 8.1. The proof is somewhat technical, requiring use of the scattering

[10] It may seem more intuitive to localize $y'$ and $y''$ separately rather than to localize $d_{\partial M}(y', y'')$, but as it turns out we cannot afford to let this cutoff be non-constant on the diagonal $z' = z''$ and so we must proceed in this fashion.

[11] In fact, only finitely many $m$ are necessary.

pseudo-differential calculus, and is deferred to an appendix (Section 10). We remark in the case where $M$ is asymptotically flat rather than asymptotically conic, one can avoid the need for this Lemma by choosing a more sophisticated commutant (see the remarks in Section 8).

Note that in our applications of Lemma 4.3, the symbols $a_w^{jk}(z)$ and $a_w^j(z)$ will often be derivatives of cutoff functions and of the distance function on $M \times M$, with $w$ corresponding to one variable and $z$ to the other.

It remains to prove Proposition 4.1 and Proposition 4.2. This will be done in the next few sections. We first however record a number of quantities which are bounded by $Z$.

**Lemma 4.4.** *We have the estimates*

$$\int_0^1 \int_{M\times M} (|\nabla u(t,z')|^2 |u(t,z'')|^2 + |u(t,z')|^2 |\nabla u(t,z'')|^2) \times (\frac{1}{\langle z'\rangle^2} + \frac{1}{\langle z''\rangle^2})\, dg(z')\, dg(z'') dt = O(Z) \tag{4.7}$$

$$\int_0^1 \int_{\langle z'\rangle, \langle z''\rangle \geq r_0} |\not\nabla u(t,z')|^2 |u(t,z'')|^2 \frac{1}{r'}\, dg(z')\, dg(z'') dt = O(Z) \tag{4.8}$$

$$\int_0^1 \int_{M\times M} |u(t,z')|^2 |u(t,z'')|^2 (1 + \frac{1}{d_M(z',z'')^2})\, dg(z')\, dg(z'') dt = O(Z). \tag{4.9}$$

$$\int_0^1 \int_{M\times M} \frac{a^{jk}(z',z'') \not\nabla_j u(t,z') \overline{\nabla_k u(t,z')}}{r'} |u(t,z'')|^2 \ dg(z') dg(z'') \ dt = O(Z) \tag{4.10}$$

$$\int_0^1 \int_{M\times M} \frac{a^{jk}(z',z'') \not\nabla_j u(t,z') \overline{u(t,z')}}{r'} u(t,z'') \overline{\nabla_k u(t,z'')} dg(z') dg(z'') \ dt = O(Z). \tag{4.11}$$

*whenever $a^{jk}(z',z'')$ are symbols of order 0 in $z'$ uniformly in $z''$ that are supported on the scattering region $\{\langle z'\rangle \geq r_0\}$ for $r_0$ sufficiently large.*

*Proof.* We begin with (4.7). Consider for instance the term

$$\int_0^1 \int_{M\times M} |\nabla u(z')|^2 |u(z'')|^2 \frac{1}{\langle z'\rangle^2}\, dg(z')\, dg(z'') dt.$$

Applying (1.11) in the $z'$ variable and (3.3) in the $z''$ variable we can bound this expression by $O(\|u(0)\|^2_{H^{1/2}(M)} \|u(0)\|^2_{L^2(M)})$, which by (3.3) again is $O(2^k \|u(0)\|^4_{L^2(M)})$ $F = O(Z)$. Now consider the term

$$\int_0^1 \int_{M\times M} |u(z')|^2 |\nabla u(z'')|^2 \frac{1}{\langle z'\rangle^2}\, dg(z')\, dg(z'') dt.$$

Now we use (1.12) in the $z'$ variable and (3.4) in the $z''$ variable to bound this by $O(\|u(0)\|^2_{H^{-1/2}(M)} 2^{2k} \|u(0)\|^2_{L^2(M)})$, which by (3.3) is again $O(2^k \|u(0)\|^4_{L^2}) = O(Z)$.

The estimate (4.8) is proven similarly to (4.7), but with (1.13) used instead of (1.11).

Now consider (4.9). The integral on the region $d_M(z',z'') < \varepsilon$ is clearly bounded by $O(\varepsilon^{-2} \|f\|^4_{L^2(M)}) = O(Z)$ thanks to (3.3). When $d_M(z',z'') < \varepsilon$ we observe that

the kernel $(d_M(z', z''))^{-2}$ has marginal integrals

$$\sup_{z''} \int_{d_M(z',z'')<\varepsilon} (d_M(z', z''))^{-2}\, dg(z'), \quad \sup_{z'} \int_{d_M(z',z'')<\varepsilon} (d_M(z', z''))^{-2}\, dg(z'') \leq C\varepsilon$$

so by Schur's test we have

$$\begin{aligned} \int \int_{d_M(z',z'')<\varepsilon} &|u(t, z')|^2 |u(t, z'')|^2 (d_M(z', z''))^{-2}\; dg(z')\, dg(z'') \\ &\leq C\varepsilon \| |u(t)|^2 \|_{L^2(M)} \| |u(t)|^2 \|_{L^2(M)}. \end{aligned}$$

Integrating this in $t$ we see that this contribution to (4.9) is $O(\varepsilon \int_0^1 \int_M |u(t, z)|^4\, dgdt) = O(Z)$ as desired.

Now we prove (4.10). From (3.3) we have

$$\int |u(t, z'')|^2\; dg(z'') = O(\|u(0)\|^2_{L^2(M)})$$

so we can bound the left-hand side of (4.10) as

$$O\big(\|u(0)\|^2_{L^2(M)} \int_0^1 \sup_{z'' \in M} \big| \int_M \frac{a^{ij}(z', z'')\nabla_i u(t, z')\overline{\nabla_j u(t, z')}}{(r')^2}\; dg(z')\big| dt\big)$$

which by Lemma 4.3 (letting the index set $W$ be the manifold $M$) and (3.3) is $O(\|u(0)\|^2_{L^2(M)} \|u(0)\|^2_{H^{1/2}(M)}) = O(2^k \|u(0)\|^4_{L^2(M)}) = O(Z)$ as desired.

Finally we prove (4.11). From (3.3), (3.4) and Cauchy-Schwarz we have

$$\int |u(t, z'')| |\nabla u(t, z'')|\; dg(z'') = O(\|u(0)\|_{L^2(M)} \|u(0)\|_{H^1(M)}) = O(2^k \|u(0)\|^2_{L^2(M)})$$

so by using Lemma 4.3 and (3.3) as with (4.10) we can bound the left-hand side of (4.11) as

$$O(2^k \|u(0)\|^2_{L^2(M)} \|u(0)\|^2_{L^2(M)}) = O(Z)$$

as desired. □

## 5. $L^4$ ESTIMATE IN THE NEAR REGION

In this section we prove the near region estimate Proposition 4.1, which is significantly easier to prove (and indeed also follows from earlier work in [20], [2]) but already illustrates the basic method. We fix $\eta$ and $z_0$, and allow our constants to depend implicitly on $z_0$, $\eta$.

Let $A_{\mathrm{near}}$ be the commutant defined in (4.3), and let $U = u \otimes u$ as above. From (2.6), Cauchy-Schwarz, (3.3), (3.4) and the trivial observation that $d_M$ is Lipschitz, we see that

$$\begin{aligned} |\langle A_{\mathrm{near}} U(t), U(t) \rangle_{M \times M}| &\leq C \|U(t)\|_{H^1(M\times M)} \|U(t)\|_{L^2(M \times M)} \\ &\leq C 2^k \|u(0)\|^4_{L^2(M)} = O(Z) \end{aligned}$$

for $t = 0, 1$. Thus by (2.1) we have

$$\int_0^1 \langle i[H_{M\times M}, A_{\mathrm{near}}] U(t), U(t) \rangle_{M\times M}\; dt \leq O(Z). \tag{5.1}$$

The idea is now to decompose the left-hand side of (5.1) as a positive main term that will yield Proposition 4.1, plus some error terms which are either positive or $O(Z)$.

We first expand the commutator using (4.3) as

$$i[H_{M\times M}, A_{\mathrm{near}}] = -[H_{M\times M}, \varphi][H_{M\times M}, d_M(z', z'')] - \varphi[H_{M\times M}, [H_{M\times M}, d_M(z', z'')]]. \tag{5.2}$$

Consider the contribution of the first term $[H_{M\times M}, \varphi][H_{M\times M}, d_M(z', z'')]$ in (5.2) to (5.1). This is an error term and will be bounded in absolute value. By integration by parts, we can bound this contribution by

$$O(\int_0^1 |\langle [H_{M\times M}, d_M]U(t), [H_{M\times M}, \varphi]U(t)\rangle_{M\times M}|\ dt). \tag{5.3}$$

To control this expression we apply (2.2) to expand

$$[H_{M\times M}, d_M(z', z'')] = \langle \nabla_{z'} d_M, \nabla_{z'}\rangle_g + \langle (\nabla_{z''} d_M), \nabla_{z''}\rangle_g + (H_{z'} d_M) + (H_{z''} d_M). \tag{5.4}$$

To estimate these quantities, we first recall the form of the Laplacian in local polar co-ordinates:

**Lemma 5.1.** *Let $x^i$ be normal co-ordinates on a Riemannian manifold $X$, centred at $p \in X$, and let $(s, \phi)$ be polar co-ordinates with respect to $x^i$. That is, $s^2 = \sum_i (x^i)^2$ and $\phi$ are homogeneous of degree zero with respect to $x^i$. Then the Laplacian takes the form*

$$\Delta_X = -\frac{\partial^2}{\partial s^2} - (1 + c)\frac{n-1}{s}\frac{\partial}{\partial s} + s^{-2}\Delta_{S^{n-1}_s}.$$

*Here $c$ is a smooth function which is $O(s^2)$ and $\Delta_{S^{n-1}_{s_0}}$ is the Laplacian on $S^{n-1}$ determined by the co-ordinates $\phi$ and the metric $s_0^{-2}g$ restricted to the submanifold $\{s = s_0\}$.*

*Proof.* The Laplacian in Riemannian polar normal co-ordinates is given by

$$\frac{1}{\sqrt{g}}\,\partial_s \sqrt{g}\,\partial_s + \frac{1}{\sqrt{g}}\partial_{\phi^i}\sqrt{g}\,g^{ij}\partial_{\phi^j}, \quad g = ds^2 + k_{ij}(s, \phi)d\phi^i d\phi^j, \quad k_{ij} = O(s^2).$$

Since in normal co-ordinates the metric is Euclidean to second order at the origin, $\sqrt{g} = s^{n-1}(1 + O(s^2))$. The result follows readily from these facts. ☐

From this and (5.4) we see that in the region $z', z'' \in B(z_0, \eta)$ we can write

$$i[H_{M\times M}, d_M(z', z'')] = O(1)\nabla_{z',z''} + O(1)/d_M(z', z'')$$

where $O(1)$ denotes various bounded functions of $z', z''$. Applying this estimate, as well as using (2.2) to expand $i[H_{M\times M}, \varphi]$, we can thus bound (5.3) by

$$O(\int_0^1 \int_{B(z_0,2\eta)\times B(z_0,2\eta)} (|U(t, z', z'')| + |\nabla_{z',z''}U(t, z', z'')|) (|U(t, z', z'')|/d_M(z', z'') + |\nabla_{z',z''}U(t, z', z'')|)\ dg(z')dg(z'')dt.)$$

Expanding out $U$ and using Cauchy-Schwarz, we can bound this by a linear combination of (4.7), (4.9). Thus these terms are $O(Z)$.

Comparing the above estimates with (5.1) and (5.2), we now have

$$\int_0^1 \langle -\varphi[H_{M\times M}, [H_{M\times M}, d_M]]U(t), U(t)\rangle_{M\times M}\ dt \le O(Z). \tag{5.5}$$

Next, we apply (2.3) to obtain

$$-[H_{M\times M},[H_{M\times M},d_M]] = -\nabla_\beta \mathrm{Hess}(d_M)^{\alpha\beta}\nabla_\alpha - H^2_{M\times M} d_M \tag{5.6}$$

where the gradients $\nabla$ are on the product manifold $M \times M$ (so the indices $\alpha, \beta$ range over six values).

Now consider the contribution of the lower order term $H^2_{M\times M} d_M(z', z'')$ to (5.5). We need

**Lemma 5.2.** *For $z', z'' \in B(z_0, 2\eta)$, we have*

$$-H^2_{M\times M} d_M(z', z'') = 8\pi\delta_{z'}(z'') + O(s^{-1}).$$

*Proof.* We again use polar normal co-ordinates. Choose a co-ordinate system so that for each fixed $z''$ the $z'$ co-ordinates are normal centred at $z''$. Then changing to polar co-ordinates $s, \phi$ in the $z'$ variable we see that

$$\begin{aligned}\Delta_{z'} d_M(z', z'') &= \Big( -\frac{\partial^2}{\partial s^2} - (1+c)\frac{n-1}{s}\frac{\partial}{\partial s} + s^{-2}\Delta_{S_r^{n-1}}\Big) s \\ &= (1+c)\frac{n-1}{s},\end{aligned}$$

where $c(z', z'')$ vanishes to second order at $z' = z''$. This condition is symmetric under interchanging $z'$ and $z''$ so the same is true of $\Delta_{M\times M} d_M(z', z'')$; specializing to $n = 3$, we thus have

$$\Delta_{M\times M} d_M(z', z'') = (4 + \tilde{c}) d_M(z', z'')^{-1}$$

where $\tilde{c}$ also vanishes to second order at $z' = z''$. Now we apply $\Delta_{M\times M}$ again using Lemma 5.1 to obtain

$$\Delta_{M\times M}\Delta_{M\times M} d_M(z', z'') = -32\pi\delta_{z'}(z'') + O(d_M(z', z'')^{-1})$$

and the claim follows (note that this is consistent with (2.9)). $\square$

Thus the contribution of this term to (5.5) is of the form

$$\begin{aligned}&8\pi \int_0^1 \int_M \varphi(z, z)|u(t, z)|^4 \, dg(z) \\ &+ O\Big(\int_0^1 \int_{B(z_0,2\eta)\times B(z_0,2\eta)} |u(z'')|^2 |u(z')|^2 / d_M(z, z') \, dg(z') dg(z'')\Big).\end{aligned}$$

The first term is the main term (as in (2.9) and (2.10)). The error term is $O(Z)$ by (4.9).

The remaining term to consider in (5.5) is the contribution of $-\nabla \mathrm{Hess}(d_M)\nabla$, which we can write using (1.5) and integration by parts as

$$\begin{aligned}&\mathrm{Re} \int_0^1 \int_{M\times M} \varphi \mathrm{Hess}(d_M)^{\alpha\beta} \nabla_\alpha U(t) \overline{\nabla_\beta U(t)} \, dg(z') dg(z'') dt \\ &+ \mathrm{Re} \int_0^1 \int_{M\times M} (\nabla_\beta \varphi) \mathrm{Hess}(d_M)^{\alpha\beta} \nabla_\alpha U(t) \overline{U(t)} \, dg(z') dg(z'') dt.\end{aligned} \tag{5.7}$$

The first term in (5.7) is the most interesting one. We use the characterization of $\mathrm{Hess}(d_M)(v, v)$, where $v = (v', v'') \in T_{(z', z'')} M^2$, as the second derivative of

$d_M(z'(t), z''(t))$ where $(z'(t), z''(t))$ moves along a geodesic in $M^2$ with initial condition $((z', z''); v)$. We use the second variation formula for geodesics (see [12], Theorem 4.1.1) to obtain

$$\begin{aligned} \frac{d^2}{dt^2} d_M(z'(t), z''(t)) &\geq -\frac{1}{\mathrm{dist}(z', z'')} \int_0^1 \left\langle R(\partial_s, \partial_t^\perp)\partial_t^\perp, \partial_s \right\rangle ds \\ &\geq -C d_M(z'(t), z''(t))\big(|v'|^2 + |v''|^2\big), \end{aligned} \tag{5.8}$$

where $\partial_s$ is the velocity of the geodesic which goes from $z'(0)$ to $z''(0)$ in unit time, hence $|\partial_s| = d_M(z', z'')$ and $\partial_t$ is the Jacobi field corresponding to the variation of the geodesic. In particular this bound does not blow up when $d_M(z', z'')$ is small. From this it follows that

$$\begin{aligned} \mathrm{Re} \int_0^1 \int_{M\times M} &\varphi \mathrm{Hess}(d_M)^{\alpha\beta} \nabla_\alpha U(t) \overline{\nabla_\beta U(t)}\ dg(z') dg(z'') dt \\ &\geq -C \int_0^1 \int_{B(z_0, 2\eta)\times B(z_0, 2\eta)} |\nabla U(t)|^2\ dg(z') dg(z'') dt. \end{aligned}$$

The right-hand side is then $O(Z)$ by (4.7).

Finally, consider the second term in (5.7). Using the crude estimate $|\mathrm{Hess}(d_M)| \leq C/d_M$ we can bound this contribution by

$$O\Big(\int_0^1 \int_{B(z_0, 2\eta)\times B(z_0, 2\eta)} |U(t, z', z'')||\nabla_{z', z''} U(t, z', z'')|/d_M(z', z'')\ dg(z') dg(z'') dt\Big).$$

By Cauchy-Schwarz we can bound this a linear combination of (4.7) and (4.9), and so these terms are also $O(Z)$.

Inserting all of the above estimates into (5.5), we obtain Proposition 4.1.

## 6. The geometry of the scattering region

To prove Theorem 1.5, it remains to verify the scattering region $L^4$ estimate in Proposition 4.2. To do this we shall need to understand the geometry of the scattering region. More precisely, we will need to control the metric function $d_M(z', z'') = d_M((x', y'), (x'', y''))$ and its derivatives on the support of $\chi\psi$, where $\chi$, $\psi$ are as in (4.4). To avoid singularities forming in $d_M$, we will always assume in this section that $r_0$ is sufficiently large and $\eta$ sufficiently small.

We begin with a basic symbol estimate on the metric $d_M$.

**Proposition 6.1.** *Let $(z', z'')$ be in the support of $\chi\psi$. If $d_M(z', z'') \geq \frac{r'+r''}{100}$, then we have the product symbol estimates*

$$|\nabla_{z'}^{m'} \nabla_{z''}^{m''} \nabla_{z', z''} d_M(z', z'')| \leq C_{m', m''} (r')^{-m'} (r'')^{-m''}$$

*for all $m', m'' \geq 0$. If instead $d_M(z', z'') \leq \frac{r'+r''}{100}$, then we have*

$$|\nabla_{z', z''}^m d_M(z', z'')| \leq C_m d_M(z', z'')^{1-m}$$

*for all $m \geq 0$. Furthermore, for $j \leq 3$, the distribution $\nabla_{z', z''}^j d_M(z', z'')$ has no mass on the diagonal $z' = z''$.*

This symbol estimate, while quite plausible, is a little technical and its proof will be deferred to Section 9. From this, Lemma 5.2, and a rescaling argument we can obtain the following estimate for the bi-Laplacian of $d_M$:

**Corollary 6.2.** *The function $H^2_{M\times M} d_M(z',z'')$ satisfies the distributional inequality*

$$-H^2_{M\times M} d_M(z',z'') \leq 8\pi\delta_{z'}(z'') - Cd_M(z',z'')^{-1}(r'+r'')^{-2} - C(r')^{-3} - C(r'')^{-3}$$

*uniformly on the support of $\chi\psi$.*

*Proof.* Observe that if $|r''/r'-1| > c$ for some $c > 0$ $d_M(z',z'')$ is comparable to $r'+r''$ and the estimate follows from Proposition 6.1, so we may assume that $|r''/r'-1| < c$ for some small $c$. In this region Proposition 6.1 is insufficient (it gives a bound of $O(d_M^{-3})$, which is not even locally integrable near the diagonal). Instead, we we set $\tau := 1/r'$ and consider the rescaled manifold[12] $M_\tau$ with metric

$$g_\tau = dr^2 + r^2 h_{ij}(\frac{\tau}{r}, y) dy^i dy^j.$$

Observe that $g_\tau$ various smoothly in $\tau$ as $\tau$ varies in the compact set $[0, 2/r_0]$. Also, since $d_{\partial M}(y',y'') \leq 2\eta$ and $|r''/r'-1| < c$, we see that

$$d_{M_\tau}(\tau z', \tau z'') \leq C(c+\eta),$$

with the action $\tau z' := (\tau r', y')$ and likewise in $z''$. Thus if $c$ and $\eta$ are sufficiently small, we can apply Lemma 5.2 and conclude that

$$-H^2_{M_\tau\times M_\tau} d_{M_\tau}(z',z'') \leq 8\pi\delta_{\tau z'}(\tau z'') - Cd_{M_\tau}(\tau z', \tau z'')^{-1}$$

uniformly in $\tau$. The claim follows by undoing the scaling. □

Finally, we need a geodesic convexity estimate on the metric $d_M(z'(s), z''(s))$, generalizing (2.8).

**Proposition 6.3.** *Let $z'(s), z''(s)$ be two geodesics. For $r_0$ sufficiently large and $\eta$ sufficiently small, the estimate*

$$\begin{aligned}\frac{d^2}{ds^2} d_M(z'(s), z''(s)) \geq &-C\Big(\frac{|v'_{\mathrm{ang}}|^2}{r'} + \frac{|v''_{\mathrm{ang}}|^2}{r''}\Big) \\ &-C\Big(\frac{|v'|^2}{(r')^2} + \frac{|v''|^2}{(r'')^2}\Big)\end{aligned} \tag{6.1}$$

*holds whenever $z'(s), z''(s)$ are distinct points in the support of $\chi\psi$, where $|v'| := |\frac{d}{ds}z'(s)|_{g(z'(s))}$ is the speed of $z'$ and $|v'_{\mathrm{ang}}| := r'(s)|\frac{d}{ds}y'(s)|_{h(z'(s))}$ is the angular component of this speed, and similarly for $v''$.*

This proposition is somewhat delicate and will be also deferred to Section 9. The idea is to first prove this lemma for perfectly conic metrics (in which the second group of terms in (6.1) do not appear); the perfectly conic metric contains a large number of totally geodesic planes on which the metric behaves like the Euclidean metric except in normal directions (generating the first group of terms in (6.1)). We then handle the approximately conic case by obtaining symbol estimates for the error between the approximately conic and perfectly conic metrics giving rise to the second group of terms in (6.1).

[12]Strictly speaking, this rescaling is only well-defined on the scattering region of the manifold, but that is all we shall need here.

## 7. $L^4$ ESTIMATE IN THE SCATTERING REGION

We can now begin the proof of Proposition 4.2. Fix $r_0, \eta$; we assume $r_0$ to be large enough and $\eta$ small enough that the geometrical lemmas of the previous section apply, and allow our constants $C$ to depend on $r_0$, $\eta$. Let $A_{\mathrm{asympt}}$ be as in (4.4). By arguing as in Section 5 we have

$$\int_0^1 \langle i[H_{M\times M}, A_{\mathrm{asympt}}]U(t), U(t)\rangle_{M\times M}\ dt \leq O(Z). \tag{7.1}$$

Again, we will expand the left-hand side of (7.1) to extract a positive main term that will give the estimate in Proposition 4.2, plus a number of error terms which are either positive or $O(Z)$.

From (4.4) we may decompose

$$\begin{aligned} i[H_{M\times M}, A_{\mathrm{asympt}}] = &-[H_{M\times M}, \chi]\psi[H_{M\times M}, d_M] \\ &-\chi[H_{M\times M}, \psi][H_{M\times M}, d_M] \\ &-\chi\psi[H_{M\times M}, [H_{M\times M}, d_M]]. \end{aligned} \tag{7.2}$$

The first two terms are error terms and will be estimated in absolute value; it is the third one which will generate the interesting positive terms.

We first analyze the first term $[H_{M\times M}, \chi]\psi[H_{M\times M}, d_M]$ in (7.2). Integrating by parts we see that the contribution of this term to (7.1) is

$$O(\int_0^1 |\langle \psi i[H_{M\times M}, \chi]U(t), i[H_{M\times M}, d_M]U(t)\rangle_{M\times M}|\ dt). \tag{7.3}$$

This integral is supported in the region where either $\langle z'\rangle \leq r_0$ or $\langle z''\rangle \leq r_0$, since $\chi$ is constant elsewhere; without loss of generality it suffices to consider the terms when $\langle z'\rangle \leq r_0$. To control this expression, we first observe from (2.2) and Lemma 6.1 that

$$i[H_{M\times M}, d_M] = O(1)\nabla_{z',z''} + O(1)/d_M(z', z'') + O(1) \tag{7.4}$$

on the support of $\chi\psi$, where we use $O(1)$ to denote various bounded (tensor-valued) functions of $z', z''$; for future reference we also remark that these $O(1)$ errors obey symbol estimates when $d_{\partial M}(y', y'') \geq \eta/2$, since in this case $d(z', z'') \geq c_\eta(r' + r'')$.

From (7.4), (2.2) and Cauchy-Schwarz, we thus see that (7.3) can be bounded in magnitude by

$$\begin{aligned} O(\int_0^1 \int_{\langle z'\rangle \leq r_0} &|\nabla u(t, z')|^2 |u(t, z'')|^2 + |u(t, z')|^2 |\nabla u(t, z'')|^2 \\ &+ |u(t, z')|^2 |u(t, z'')|^2 (1 + d_M(z', z'')^{-2})\ dg(z') dg(z'')). \end{aligned}$$

The first two terms are $O(Z)$ by (4.7). For the third term, observe that the contribution of the region $r'' < 4r_0$ is $O(Z)$ by (4.9), while for the region $r'' > 4r_0$ we simply estimate $(1 + d_M(z', z'')^{-2})$ crudely by $O(1)$, and this term is then bounded using (3.3) by $O(\|u(0)\|^4_{L^2(M)}) = O(Z)$. Thus the total contribution of the first term of (7.2) to (7.1) is $O(Z)$.

Now consider the contribution of the second summand of (7.2) to (7.1); these are also error terms, but of a more delicate nature. Again we integrate by parts to estimate this contribution by

$$O(\int_0^1 |\langle i[H_{M\times M}, \psi]\chi U(t), i[H_{M\times M}, d_M]U(t)\rangle_{M\times M}|\ dt). \tag{7.5}$$

The key point here is that since the angular cutoff $\psi$ does not depend on the radial variables $r', r''$, the first-order operator $[H_{M\times M}, \psi]$ only contains angular derivatives $\nabla_{y',y''}$ and no radial derivatives $\nabla_{r',r''}$, thanks to (2.2). Indeed, on the support of $\chi$ we have

$$\begin{aligned}[H_{M\times M}, \psi] =& O(1)\frac{1}{(r')^2}\nabla_{y'} + O(1)\frac{1}{(r'')^2}\nabla_{y''} \\ &+ \frac{1}{(r')^2}O(1) + \frac{1}{(r'')^2}O(1)\end{aligned}$$

where again we use $O(1)$ to denote various bounded functions of $z', z''$ which also obey symbol estimates. From this, (7.4), and symmetry we see that these contributions are bounded by a combination of expressions of the form

$$O\Big(\Big|\int_0^1 \int_{M\times M} \frac{a^{jk}(z',z'')\nabla\!\!\!/_j u(t,z')\overline{\nabla\!\!\!/_k u(t,z')}}{r'}|u(t,z'')|^2\ dg(z')dg(z'')\ dt\Big|\Big),$$

$$O\Big(\Big|\int_0^1 \int_{M\times M} \frac{a^{jk}(z',z'')\nabla\!\!\!/_j u(t,z')\overline{u(t,z')}}{r'}u(t,z'')\overline{\nabla\!\!\!/_k u(t,z'')}dg(z')dg(z'')\ dt\Big|\Big),$$

$$O\Big(\Big|\int_0^1 \int_{r',r''\geq r_0} \frac{|\nabla u(t,z')||u(t,z')||u(t,z'')|^2}{r' d_M(z',z'')}\ dg(z')dg(z'')\ dt\Big|\Big),$$

$$O\Big(\Big|\int_0^1 \int_{r',r''\geq r_0} \frac{|\nabla u(t,z')||u(t,z')||u(t,z'')|^2}{(r')^2}\ dg(z')dg(z'')\ dt\Big|\Big),$$

$$O\Big(\Big|\int_0^1 \int_{r',r''\geq r_0} \frac{|u(t,z')|^2|u(t,z'')|^2}{(r')^2 d_M(z',z'')}\ dg(z')dg(z'')\ dt\Big|\Big)$$

(noting that $\nabla_y = r\nabla\!\!\!/$), where $a^{jk}$ denotes symbols supported on the the support of $\chi\nabla\psi$ (note that we do not encounter the singularity on the diagonal $z' = z''$ because $\nabla\psi$ vanishes here). The first two terms are controlled by (4.10), (4.11) respectively. The third and fourth terms can be controlled via Cauchy-Schwarz by a linear combination of (4.7) and (4.9). Finally, the last term is easily controlled by (4.9). (Recall that $r'$ is large.) Thus the total contribution of the second term of (7.2) to (7.1) is $O(Z)$.

In light of (7.1), (7.2), and the preceding estimates, we see that

$$\int_0^1 \langle -\chi\psi[H_{M\times M}, [H_{M\times M}, d_M(z',z'')]]U(t), U(t)\rangle_{M\times M}\ dt \leq O(Z). \tag{7.6}$$

Of course, we may expand $-[H_{M\times M}, [H_{M\times M}, d_M(z',z'')]]$ using (2.3) as

$$-[H_{M\times M}, [H_{M\times M}, d_M(z',z'')]] = -\nabla_\beta \mathrm{Hess}(d_M)^{\alpha\beta}\nabla_\alpha - H^2_{M\times M} d_M \tag{7.7}$$

where $\mathrm{Hess}(d_M)^{\alpha\beta} = \nabla^\alpha\nabla^\beta d$.

Let us first study the lower order term $-H^2_{M\times M} d_M(z',z'')$. Applying Corollary 6.2, the contribution of this term to (7.6) is greater than or equal to

$$\begin{aligned}&\geq 8\pi \int_0^1 \int_M \chi(z,z)^2|u(t,z)|^4\, dg(z)\, dt \\ &+O(\int_0^1 \int_{M\times M} \frac{|u(t,z')|^2|u(t,z'')|^2}{d_M(z',z'')(r'+r'')^2}\ dg(z')dg(z'')dt).\end{aligned}$$

The error term here is of course $O(Z)$ by (4.9).

It remains to treat the contribution of the $-\nabla \mathrm{Hess}(d_M)\nabla$ to (7.6). Again we integrate by parts, creating a main term

$$(7.8)\quad \mathrm{Re}\int_0^1 \int_{M\times M} \langle \chi(z',z'')\psi(y',y'')\mathrm{Hess}(d_M)^{\alpha\beta}\nabla_\alpha U(t,z',z'')\overline{\nabla_\beta U(t,z',z'')}\; dt$$

and an error term which is bounded by

$$\int_0^1 \int\limits_{M\times M} |\nabla_{z',z''}(\chi(z',z'')\psi(y',y''))||\mathrm{Hess}(d_M)||\nabla_{z',z''}U(t,z',z'')||U(t,z',z'')|\; dt.$$

Consider the error term first. For this term we use the crude estimate

$$|\mathrm{Hess}(d_M)(z',z'')| \le C(1+d_M(z',z'')^{-1})$$

from Lemma 6.1, as well as

$$|\nabla_{z',z''}(\chi(z',z'')\psi(y',y''))| \le \frac{1}{r'}+\frac{1}{r''}.$$

By symmetry we can thus estimate this error term by

$$C\int_0^1 \int_{M\times M} |\nabla u(t,z')||u(t,z')||u(t,z'')|^2(\frac{1}{r'}+\frac{1}{r''})(1+\frac{1}{d_M(z',z'')})\; dg(z')dg(z'')dt.$$

By Cauchy-Schwarz one can estimate this by a linear combination of (4.7) and (4.9).

Now consider the main term (7.8). Recall that $\mathrm{Hess}(d_M)(v,v)$, where $v \in T_{(z',z'')}M^2$, is the second derivative of $d_M(z'(s),z''(s))$ in $s$ as $(z'(s),z''(s))$ moves along a geodesic in $M^2$ with initial condition $((z',z'');v)$. By Proposition 6.3, we see that

$$\begin{aligned}(7.8) \ge &-C\int_0^1 \int_{r',r''\ge r_0/2} \Big(\frac{|\nabla\!\!\!/ u(t,z')|^2|u(t,z'')|^2}{r'}+\frac{|u(t,z')|^2|\nabla\!\!\!/ u(t,z'')|^2}{r''}\Big)\\ &+\Big(\frac{|\nabla u(t,z')|^2|u(t,z'')|^2}{(r')^2}+\frac{|u(t,z')|^2|\nabla u(t,z'')|^2}{(r'')^2}\Big)\; dg(z')dg(z'')dt.\end{aligned}$$

But the first term on the right-hand side is $O(Z)$ by (4.8), while the second term is also $O(Z)$ by (4.7). Combining all of these estimates together we obtain Proposition 4.2 (once we verify Proposition 9.4 and Proposition 6.3 in the next section). This completes the proof of Theorem 1.5.

## 8. Concluding remarks

We first want to comment on some particular geometric settings in which the proof of the Morawetz estimate is much easier. The first is that of a simply-connected smooth three-manifold $M$ which has globally non-positive, bounded sectional curvature; by the Hadamard-Cartan theorem (see [12], Corollary 4.8.1), such a manifold is diffeomorphic to $\mathbf{R}^3$. Such a manifold is automatically non-trapping, and the distance function $d_M(z',z'')$ is smooth on all of $M\times M$ outside the diagonal. We also make the technical assumption that the distance function $d_M(z',z'')$ is uniformly $C^4$ in the off-diagonal region $\{(z',z'')\in M\times M : d_M(z',z'')>1\}$[13]. In particular $H^2 d_M$ is bounded in this region. More importantly the distance function is globally convex (see [12], Theorem 4.1.1), so that the tensor $\mathrm{Hess}(d_M)^{\alpha\beta}$ is positive semidefinite, by the second variation formula. Thus with the commutant

[13]It may be that this is automatically true, but this is not clear to us.

$A = i[H, d_M(z', z'')]$ (with no cutoffs whatsoever) we obtain a Morawetz inequality very easily for this class of manifolds.

The next case we consider is that of asymptotically flat three-dimensional non-trapping manifolds, i.e. where $M = \mathbf{R}^3$ and the metric $g$ satisfies (1.1) with $h$ equal to the standard metric on $S^2$. We make a distinction between the distance $d(z', z'')$ on $M$ induced by the metric $g$ and the Euclidean distance $|z' - z''|$ obtained by identifying $M$ with Euclidean space. We shall only discuss the Morawetz estimate in the scattering region $\langle z' \rangle, \langle z'' \rangle \geq r_0$. Here the key observation is that when $z'$ and $z''$ are fairly close together, e.g. $|z' - z''| < \frac{1}{2}|z'|$, then the distance $d_M(z', z'')$ is smooth and is close to $|z' - z''|$ in the sense that

$$\left| d_M(z', z'') - |z' - z''| \right| \leq C \text{ whenever } \langle z' \rangle, \langle z'' \rangle \geq r_0 \text{ and } |z' - z''| < \frac{1}{2}|z'|,$$

at least if we make the compact region $K_0$ sufficiently large. This is an easy consequence of the decay of the difference $g - \delta$ between the two metrics. In fact, the error obeys symbol estimates of one order better than Proposition 6.1 suggests; see Proposition 9.4 in the appendix. We now consider the commutant $A := i[H, a]$, where $a$ is the function

$$a := \chi(z', z'')(\varphi(z', z'') d_M(z', z'') + (1 - \varphi(z', z''))|z' - z''|), \tag{8.1}$$

$\varphi$ is a smooth cutoff which equals 1 when $|z' - z''| \leq \frac{1}{4}|z'|$ and equals 0 when $|z' - z''| \geq \frac{1}{2}|z'|$, and $\chi$ is as in Section 4. Thus this cutoff is equal to the actual distance $d(z', z'')$ when $z'$ and $z''$ are close, but reverts smoothly to the Euclidean distance when $z'$ and $z''$ are far apart. This significantly reduces the error caused by differentiating the cutoff $\varphi$, indeed it generates terms that are now controlled by (4.7) rather than the terms (4.9), (4.10), (4.11) which required the Morawetz estimate (10.6) and its refinement in Lemma 4.3. We omit the details.

Next, we give an example to show that (4.5) cannot be strengthened by bringing the absolute value signs inside the $z$-integral.

**Proposition 8.1.** *There does not exist an estimate of the form*

$$\int_0^1 \int_M \frac{\left| \not\nabla u(t, z) \right| \left| \nabla u(t, z) \right|}{r} \, dg \, dt \leq C \|u(0)\|^2_{H^{1/2}(M)} \qquad \textit{(FALSE!)} \tag{8.2}$$

*for solutions to* (1.2) *even in Euclidean space* $(M, g) = (\mathbf{R}^3, \delta)$.

*Proof of falsity of* (8.2). We give a sketch only. Pick a large integer $N = 2^k > 1$, and let $\phi$ be the explicit solution to (1.2) given by

$$\phi(t, z) := (t + i)^{-3/2} e^{\frac{i|z|^2}{2(t+i)}};$$

note that at time $t = 0$ this is basically a standard Gaussian, while for times $t \gg 1$, the solution $\phi(t)$ is concentrated on a ball of radius $O(t)$ centered at the origin, and it (and its derivatives) have magnitude roughly $t^{-3/2}$ on this ball. We then let $u$ be the function

$$u(t, z) := k^{1/2} \phi(N^2 t, N z) + \sum_{j=1}^{k} \epsilon_j \phi(N^2 t, N(z - 2^j e_1))$$

where $e_1 = (1, 0, 0)$ is a standard unit vector and $\epsilon_j = \pm 1$ are random signs chosen independently; the purpose of the signs is to ensure (thanks to Khinchin's inequality) that we do not expect any unusual cancellation between the summands.

The function $u(t,z)$ is clearly a solution to (1.2), and the $H^{1/2}(\mathbf{R}^3)$ norm can be computed as

$$\|u(0)\|_{H^{1/2}(\mathbf{R}^3)} \sim k^{1/2}N^{-1};$$

this is easiest shown by first verifying the more general statement $\|u(0)\|_{H^s(\mathbf{R}^3)} \sim k^{1/2}N^{s-3/2}$ for $s = 0,1,2$ (applying a rescaling by $N$ if desired) and then interpolating. Thus the right-hand side of (8.2) is $O(kN^{-2})$.

Let us choose any $N/3 \leq j \leq 2N/3$ and consider the size of $u(t,z)$ and its derivatives in the region of spacetime where $t \sim 2^j/N$ and $|z - 2^j e_1| \ll 2^j$. In this region $|\nabla k^{1/2}\phi(N^2t, Nz)|$ has size $\sim k^{1/2}(2^jN)^{-3/2}$, and so the expectation of $|\nabla u(t,z)|$ is also at least $\gtrsim k^{1/2}(2^jN)^{-3/2}$. Since $\phi$ is radial, the angular derivative $|\not\nabla k^{1/2}\phi(N^2t, Nz)|$ vanishes, however $|\not\nabla\phi(N^2t, N(z - 2^je_1))|$ is fairly large, comparable to $\sim (2^jN)^{-3/2}$ in a large fraction of the region under consideration. Thus this region of spacetime contributes at least $\gtrsim k^{1/2}(2^jN)^{-3}(2^jN)2^{3j}/2^j \sim k^{1/2}N^{-2}$ to the integral in (8.2). Summing over all $j$ we see that the left-hand side of (8.2) is at least $\sim k^{3/2}N^{-2}$. Comparing this with the right-hand side we obtain the desired contradiction[14] by setting $k \to \infty$. $\square$

*Remark* 8.2. One can construct a similar example to show that the second estimate in Lemma 4.3 similarly fails if the absolute values are placed inside the integral.

*Remark* 8.3. Note that in our example $\not\nabla u(t,x)$ and (the largest term in) $\nabla u(t,x)$ oscillate in different directions and so if one removes the absolute values, replacing instead by a smooth symbol, then we do not obtain a counterexample to Lemma 4.3.

Finally, we remark on the applicability of the Strichartz estimate in Theorem 1.5 to non-linear Schrödinger equations. It turns out that in such equations, the most useful Strichartz estimates are either those that require no derivatives whatsoever on $u$ (i.e. they have $L^2(M)$ on the right-hand side), or measure $u$ in a space of the form $L^q_t L^\infty_x$. Our current estimate has neither of these two properties. However by commuting the Schrödinger flow with a power of $(1+H)$ and using Sobolev embedding one can get an $L^q_t L^\infty_x$ estimate, for instance we have

$$\|u\|_{L^4_t L^\infty_x([0,1]\times M)} \leq C_\varepsilon \|u(0)\|_{H^{1+\varepsilon}(M)}$$

for any $\varepsilon > 0$. This result can for instance be used to demonstrate local well-posedness in $H^{1+\varepsilon}(M)$ of the quintic non-linear Schrödinger equation $iu_t + \frac{1}{2}\Delta u = \pm|u|^4u$ on non-trapping asymptotically conic manifolds by a standard argument which we omit (see e.g. [5], [3]). Note that the above estimate was also derived by Burq [2] for asymptotically flat non-trapping manifolds.

It would be interesting to see if the $\varepsilon$ can be removed; this could then be used to demonstrate *global* well-posedness of the above quintic equation in the energy space $H^1(M)$ for small energy data, using the conservation of energy in the standard manner (note the sign of the non-linearity is irrelevant in the small energy setting). While the $\varepsilon$ can indeed be removed from the above Strichartz estimate in the Euclidean space case (see [24]; the corollary concerning global well-posedness can be achieved by many other means, see for instance [5]), we were unable to remove

[14]Note that this is only a 'logarithmic' failure in the frequency variable. Indeed if we concede an epsilon worth of derivatives then we can handle these terms easily by a variant of (1.11) where the epsilon loss is transferred from the weight $r^{-1-\varepsilon}$ to the regularity $H^{1/2+\varepsilon}(M)$, thus recovering the result of Burq [2], at least in the context of $L^4_{t,x}$ estimates.

it here. However, by a Besov space interpolation argument (see [24]) it is possible to obtain the variant estimate

$$\|u\|_{L^q_t L^\infty_x([0,1]\times M)} \leq C_q \|u(0)\|_{H^{3/2-2/(q-1)}(M)}$$

for all $4 < q < \infty$. This suffices to give local well-posedness on non-linear Schrödinger equations $iu_t + \frac{1}{2}\Delta u = F(u)$ in the space $H^{3/2-2/(q-1)}(M)$ for any $q > 4$, whenever $F(u)$ is smooth and grows like $|u|^q$; we omit the details as they are rather standard (see e.g. [5]). Note that in Euclidean space the space $H^{3/2-2/(q-1)}$ would be critical with respect to the usual scaling of the non-linear Schrödinger equation. Thus the Strichartz estimate Theorem 1.5 has some application to non-linear Schrödinger equations, though clearly it is less satisfactory than the full range (1.8) of Strichartz estimates would be; these would yield local well-posedness in $H^s$ for any $q > 1$ and $s \geq \max(0, \frac{3}{2} - \frac{2}{q-1})$.

## 9. Appendix I: Proof of geometrical lemmas

In this section we analyze the behaviour of geodesics in thin conic regions of $M$, in order to prove the geometric estimates in Proposition 6.1 and Proposition 6.3 required in the above proof of Theorem 1.5. The main strategy is to compare the asymptotically conic metric $d_M$ to a perfectly conic metric $d_{\mathrm{conic}}$, which has an exact formula and can be handled explicitly.

By compactness of $\partial M$ it suffices to work in a truncated sector

$$\Omega(y_0, \eta, r_0) := \{(r, y) \mid d_{\partial M}(y, y_0) < \eta; \langle z\rangle > r_0\} \tag{9.1}$$

for some $y_0 \in \partial M$, where we assume $r_0$ sufficiently large and $\eta$ sufficiently small (possibly depending on $y_0$, although this is irrelevant since $y_0$ ranges in a compact set).

We first briefly review geodesic flow on a general Riemannian manifold $(M, g)$ parameterized by a general set of co-ordinates $z^j$. We will view geodesic flow as a Hamiltonian flow on the cotangent bundle $T^*(M)$ of $M$, which we parameterize as $(z^j, \zeta_j)$, where $\zeta_j$ is the co-ordinate dual to the vector field $dz^j$; thus $\zeta_j dz^j$ is the canonical one-form. On this cotangent bundle we define the energy function $\sigma(H) : T^*(M) \to \mathbf{R}$ as

$$\sigma(H) := \frac{1}{2} g^{jk}(z)\zeta_j\zeta_k = \frac{1}{2}|\zeta|^2_{g(z)};$$

note that this is also the symbol of the Schrödinger operator $H$. This energy function then induces trajectories $(z(s), \zeta(s))$ in the cotangent bundle which are just the paths of geodesic flow (using the metric $g$ to identify the cotangent and tangent bundles):

$$\frac{d}{ds} z^j = g^{jk}(z)\zeta_k$$
$$\frac{d}{ds}\zeta^j = -\frac{1}{2}\frac{\partial g^{kl}}{\partial z^j}(z)\zeta_k\zeta_l.$$

These flows preserve $\sigma(H)$, and are also homogeneous with respect to the scaling $(z(s), \zeta(s)) \mapsto (z(\lambda s), \lambda\zeta(\lambda s))$, which corresponds to speeding up the velocity of the geodesic flow by $\lambda$. If we use this homogeneity to normalize to the unit speed case $\sigma(H) = \frac{1}{2}$ (i.e. $|\zeta|_{g(z)} = 1$), then this flow is related to the distance function $d_M$ on $M$ by the formula

$$d_M(z(s), z(s')) = |s - s'|$$

provided that $|s - s'|$ is sufficiently small. Thus the short-time geodesic flow can be used to define the metric locally.

Now we specialize to studying geodesic flow in the truncated sector $\Omega(y_0, \eta, r_0)$ in the scattering co-ordinates $(x, y^j)$, where the manifold $M$ is either asymptotically conic or perfectly conic. Of course this region is not geodesically complete, so the statements below are conditional assuming that the geodesic does indeed stay inside this region.

The first step is to choose co-ordinates for the cotangent bundle $T^*\Omega(y_0, \eta, r_0)$. We shall use the co-ordinates $(x, y^j, \nu, \mu_j)$, where $0 < x < \epsilon_0$, and $y \in \partial M$, while the co-ordinates $\nu$ and $\mu_j$ are dual to the vector fields $-x^2\partial_x = \partial_r$ and $x\partial_{y^j}$ respectively[15]; thus one can think of $\mu$ as an element of $T^*_y(\partial M)$, the canonical one-form is $-\nu dx/x^2 + \mu^j dy_j/x$. Following Melrose [16], we define the *scattering cotangent bundle* over the compact manifold $\overline{M}$ as the bundle ${}^{\mathrm{sc}}T^*\overline{M}$ whose sections are locally spanned over $\mathcal{C}^\infty(\overline{M})$ by $dx/x^2$ and $dy/x$ (hence can be paired with the *scattering vector fields*, spanned by $-x^2\partial_x$ and $x\partial_y$). Because of the form of the metric, the symbol $\sigma(H)$ of the Schrödinger operator $H$ now has the form

$$\sigma(H) = \frac{1}{2}(\nu^2 + h_{ij}(z)\mu^i\mu^j) = \frac{1}{2}(\nu^2 + |\mu|^2_{h(z)})$$

and thus in the unit speed case $\sigma(H) = \frac{1}{2}$, $\nu$ and $\mu$ will have combined magnitude equal to 1.

The geodesic flow on ${}^{\mathrm{sc}}T^*\overline{M}$ can then be written explicitly as

$$\begin{aligned} \frac{d}{ds}x &= -x^2\nu \\ \frac{d}{ds}y^j &= xh^{jk}\mu_k \\ \frac{d}{ds}\nu &= xh^{jk}\mu_j\mu_k + \frac{1}{2}x^2\frac{\partial h^{jk}}{\partial x}\mu_j\mu_k \\ \frac{d}{ds}\mu_j &= -x\mu_j\nu - \frac{1}{2}x\frac{\partial h^{kl}}{\partial y^j}\mu_k\mu_l; \end{aligned} \tag{9.2}$$

recall that $h^{jk}(x, y)$ is, for small $x$, a family of metrics in $y$ with $x$ a smooth parameter; it is only in the perfectly conic case that there is no $x$-dependence.

This geodesic flow of course still preserves $\sigma(H)$, and is still homogeneous with respect to the scaling $(x(s), y(s), \nu(s), \mu(s)) \mapsto (x(\lambda s), y(\lambda s), \lambda\nu(\lambda s), \lambda\mu(\lambda s))$. This geodesic flow equation can be used to define the distance $d_M((x', y'), (x'', y''))$ provided $(x', y')$ and $(x'', y'')$ are sufficiently close; we will quantify what 'sufficiently close' means later.

To gain some intuition as to how the distance $d_M$ behaves on $\Omega(y_0, \eta, r_0)$, let us first consider the case $h^{jk}(x, y) = h^{jk}(y)$ when $M$ is perfectly conic near infinity; to distinguish this from the asymptotically conic situation we shall write $d_{\mathrm{conic}} = d_{\mathrm{conic},h}$ instead of $d_M$, and write the metric $g$ as $g_{\mathrm{conic},h}$. In this case the equations for geodesic flow are equivalent to those on a plane, for the following geometric reason. Consider a geodesic $\gamma$ in $B_{\partial M}(y_0, \eta)$, and consider the truncated cone $\{(x, y) : 0 < x < \epsilon_0, y \in \gamma\}$ over that geodesic. In the perfectly conic case, this

[15]More informally, a symbol such as $a(x, y, \nu, \mu)$ corresponds in the usual polar co-ordinates $r, y$ to the pseudo-differential operator $a(\frac{1}{r}, y, \frac{\partial}{\partial r}, \not\nabla)$. The advantage of these co-ordinates is that $\nu, \mu$ remain bounded for unit speed geodesic flows.

truncated cone is both perfectly flat and totally geodesic[16], and in particular is isometric to a subset of the plane. Conversely, it is easy to show that there is a unique minimal geodesic connecting $(x', y')$ and $(x'', y'')$ which lies in the truncated cone over the geodesic in $\partial M$ connecting $y'$ and $y''$. From plane geometry we thus see that we have the *cosine rule*[17]

$$d_{\mathrm{conic}}(z', z'')^2 = (r')^2 + (r'')^2 - 2r'r'' \cos d_{\partial M}(y', y'') \tag{9.3}$$

for $z', z'' \in \Omega(y_0, \eta, r_0)$. Note in particular that we thus expect symbol-type regularity estimates on $d_{\mathrm{conic}}$, in the sense that each derivative in the $z'$ and $z''$ variable gains a power of $d$; this is of course what happens in the Euclidean case, and we will be able to show that it also happens in asymptotically conic manifolds near infinity. This is quite plausible when $r'$ and $r''$ are of comparable magnitude; the most delicate issue will be proving these sorts of bounds when $r'$ and $r''$ are both large but far apart from each other, which seems to be a geometric necessity in order to not lose any derivatives in Theorem 1.5.

This completes our discussion of the perfectly conic case. We now consider the distance function $d_M$ of an asymptotically conic manifold on $\Omega(y_0, \eta, r_0)$. We shall always take $\eta$ to be sufficiently small and $r_0$ to be sufficiently large.

We first show that globally geodesics between points in $\Omega(y_0, \eta, r_0)$ stay in the scattering region.

**Lemma 9.1.** *If $\eta$ is sufficiently small and $r_0$ sufficiently large, then for every pair of points $z', z''$ in $\Omega(y_0, \eta, r_0)$, every globally minimizing geodesic between them lies entirely inside $\Omega(y_0, 5\eta, r_0/2)$.*

*Proof.* Without loss of generality we may take $r'' \geq r'$. Suppose for contradiction that the globally minimizing curve connecting $z'$ and $z''$ contains a point $(r, y)$ in the complement of $\Omega(5\eta, r_0/2)$. Then either $r < r_0/2$ or $d_{\partial M}(y, y_0) > 5\eta$.

First suppose that $r < r_0/2$. Using the crude estimate $g \geq dr^2$, we thus obtain the lower bound

$$d_M(z', z'') \geq (r' - r_0/2) + (r'' - r_0/2) \geq r''. \tag{9.4}$$

On the other hand, if $r_0$ is large enough then we have the pointwise bound $h_{ij}(1/r, y) \leq 4h_{ij}(0, y)$ for all $r > r_0/10, y \in \partial M$, and hence $g \leq g_{\mathrm{conic},4h}$ in ths region. Since the minimal geodesic connecting $z'$ to $z''$ in the $g_{\mathrm{conic},4h}$ metric lies in the region $r > r_0/10$, we thus obtain from (9.3)

$$d_M(z', z'') \leq d_{\mathrm{conic},4h}(z', z'') \leq \sqrt{(r')^2 + (r'')^2 - 2r'r'' \cos 4\eta}, \tag{9.5}$$

which contradicts (9.4) if $\eta$ is sufficiently small.

Next suppose that $d_{\partial M}(y, y_0) > 5\eta$. Then the distance from $y$ to both $y'$ and $y''$ must be at least $4\eta$. Using the comparison $h_{ij}(0, y)/4 \leq h_{ij}(1/r, y) \leq 4h_{ij}(0, y)$, which holds for $r > r_0/2$ if $r_0$ is sufficiently large, we have the pointwise bound

[16] Indeed, one can think of $\gamma$ as being isometric to an arc in $S^1$, so the truncated cone is then isometric to a truncated sector in $\mathbf{R}^2$.

[17] This rule can also be deduced from (9.2); the point is that the $\frac{\partial h^{ij}}{\partial x}$ term in the equation for $\frac{d}{ds}\nu$ disappears in the perfectly conic case, and so the geodesic flow equation decouples into a (rescaled) geodesic flow equation on $\partial M$ for $y, \mu$, and an explicitly solvable evolution for $x$, $\nu$. We omit the details.

$dr^2 + h_{ij}(0,y)/4 \le g \le dr^2 + 4h_{ij}(0,y)$. Comparison with this smaller conic metric shows that the length of the curve must be at least

$$\sqrt{(r')^2 + r^2 - 2r'r\cos 2\eta} + \sqrt{(r^2 + (r'')^2 - 2rr''\cos 2\eta}, \tag{9.6}$$

since the broken geodesic in the small conic metric that goes between these points has at least this length. For $\eta < \pi/12$ this is larger than (9.5), as can be seen by consideration of the lengths of a plane Euclidean triangle with vertices having polar coordinates $(r', 0)$, $(r'', 4\eta)$ and $(r', 2\eta)$. This is a contradiction. □

We now use a symplectic argument to deduce regularity of the distance function $d_M(z', z'')$ where $(z', z'')$ lie in some cone $\Omega(y, \eta, r_0)$ for fixed $(\eta, r_0)$. To do this, we fully exploit the smoothness of the metric function $h_{ij}(x, y)$ in scattering coordinates $(x, y)$. We shall need to work on the a blown up version of the compactified double space $\overline{M}^2$ in order to do this.

Let $\overline{M}_b^2$ denote the blown-up compactified manifold consisting of $\overline{M}^2$ with the boundary $(\partial M)^2$ blown up in the sense of [16] (shown in Figure 1). We define $\rho = x'/x''$ and label the hypersurfaces of $\overline{M}_b^2$ by lb ('left boundary'), rb ('right boundary') and bf ('blown-up face') according as they arise from the faces $x' = 0$, $x'' = 0$ or $x' = x'' = 0$ of $\overline{M}^2$, respectively. Let $U$ be any neighbourhood in $\overline{M}_b^2$ of the closure of the diagonal $\{(z, z) : z \in M\} \cup \{w \in \mathrm{bf} : \rho(w) = 1, y'(w) = y''(w)\}$. For instance, $U$ could be the region $\{(z', z'') \in M : d_M(z', z'') < c(\langle z'\rangle + \langle z''\rangle)\}$ for some small $0 < c \ll 1$, together with the boundary of this region on the blown up face bf.

**Proposition 9.2.** *Let $\Omega(\eta, r_0) \subset \overline{M}_b^2$ be defined by*

$$\Omega(\eta, r_0) = \{x', x'' < r_0^{-1},\ d_{\partial M}(y', y'') < \eta\}$$

*Then for $r_0$ sufficiently large and $\eta$ sufficiently small,*

$$\text{the function } \frac{d_M(z', z'')}{r' + r''} \text{ is in } \mathcal{C}^\infty(\Omega(\eta, r_0) \setminus U). \tag{9.7}$$

*Remark* 9.3. Notice that $U$ includes a *conic* neighbourhood of the diagonal near infinity, that is, a set of the form

$$\{(r', y', r'', y'') \mid \big|\frac{r''}{r'} - 1\big| + d_{\partial M}(y', y'') < \epsilon\} \tag{9.8}$$

for some $\epsilon > 0$, because of the topology of $\overline{M}_b^2$.

*Proof.* To prove the claim, we use a symplectic technique; namely, we use the function $\Phi = d_M(z', z'')$ as a generating function for a Lagrangian submanifold. This function satisfies the eikonal equation

$$|\nabla_{z'}\Phi|^2 = 1, \tag{9.9}$$

and vanishes at the diagonal. Consequently the graph of $d\Phi$ will be tangent to the flow in ${}^{\mathrm{sc}}T^*(\overline{M}) \times {}^{\mathrm{sc}}T^*(\overline{M})$, lifted to a vector bundle over $\overline{M}_b^2$, generated by the energy $\frac{1}{2}(|\zeta'|_g^2 - 1)$. This flow is given by (9.2) for the singly-primed variables, while $z''$

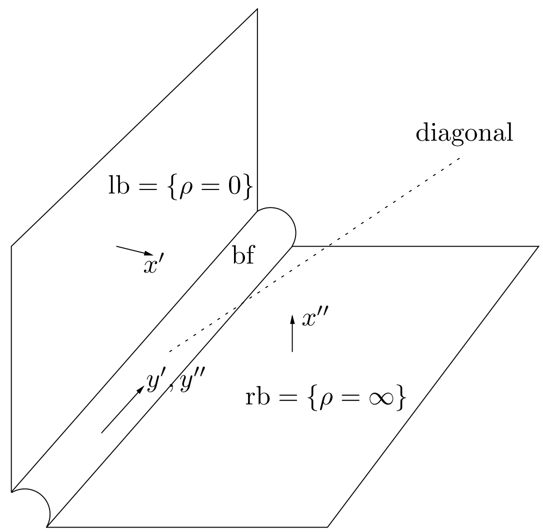


FIGURE 1. The blown up space $\overline{M}_b^2$

and $\zeta''$ are stationary under the flow. Using coordinates $(\rho, y', x'', y''; \nu', \mu', \nu'', \mu'')$, the vector field and the evolution of $\Phi$ is given by

$$
\begin{aligned}
&\frac{d}{ds}\rho = -x'\rho\nu && \frac{d}{ds}x'' = 0 \\
&\frac{d}{ds}(y')^i = x'h^{ij}\mu'_j && \frac{d}{ds}y'' = 0 \\
&\frac{d}{ds}\nu' = x'h^{ij}\mu'_i\mu'_j + \frac{(x')^2}{2}\frac{\partial h^{ij}}{\partial x'}\mu'_i\mu'_j && \frac{d}{ds}\nu'' = 0 \qquad \frac{d}{ds}\Phi = 1. \\
&\frac{d}{ds}\mu'_k = -x'\mu'_k\nu' - \frac{x'}{2}\frac{\partial h^{ij}}{\partial (y')^k}\mu'_i\mu'_j. && \frac{d}{ds}\mu'' = 0
\end{aligned}
\tag{9.10}
$$

Here $h_{ij}$ is always a function of the $(x', y')$ variables. Geometrically we consider the submanifold $L$ determined by the 'initial condition' that

$$
\{x' = x'',\ y' = y'',\ \mu' = -\mu'',\ \nu' = -\nu'',\ h^{ij}(x', y')\mu'_i\mu'_j + \nu^2 = 1\} \subset L,
\tag{9.11}
$$

and $L$ is invariant under the forward flow of (9.10) (i.e. for positive 'time'). This submanifold is a smooth Lagrangian manifold, with boundary, given by the graph of $d\Phi$, at least near the diagonal. (It is perhaps counterintuitive that $L$ is smooth near the diagonal, since $\Phi$ is singular there. The singularity of $\Phi$ is reflected in the fact that $L$ projects diffeomorphically to $\overline{M}_b^2$ near the diagonal, but not at the diagonal. The inverse image of a point on the diagonal is a $(n-1)$-sphere consisting of all possible unit directions at that point. Smoothness of $L$ is reflected in the fact that $d\Phi$ does not blow up at the diagonal, rather it becomes 'multivalued'.)

Smoothness of $L$ persists as long as the vector field of which it is the flowout remains smooth and nonvanishing. We want to show that the function $\Phi/(r' + r'')$ extends to a smooth function on $\Omega_b$ (including its boundary at infinity). To do this, we shall multiply the vector field (9.10) by a factor, which does not change the flowout. The new vector field will be smooth *and nonvanishing up to the boundary*

in suitable coordinates, and this will imply that $L$ is a smooth submanifold in these coordinates.

By symmetry of $\Phi$ under interchange of $z'$ and $z''$, it is sufficient to work in the region $\{r'' < 2r'\} = \{\rho < 2\}$. We shall refer to $\Omega(\eta, r_0) \cap \{\rho < 2\}$ as the 'region of interest'. In the region of interest it is equivalent to require that $\kappa = x'\Phi = \Phi/r'$, as opposed to $\Phi/(r' + r'')$, is smooth on $\Omega_b$. Its evolution is given by

$$\frac{d}{ds}\kappa = x'\frac{d}{ds}\Phi + \phi\frac{d}{ds}x' = x' - x'\kappa\nu'.$$

As a first step we multiply the flow (9.10) by $(x')^{-1}$, obtaining the new flow

$$\begin{aligned} &\frac{d}{ds'}\rho = -\rho\nu \\ &\frac{d}{ds'}(y')^i = h^{ij}\mu'_j \\ &\frac{d}{ds'}\nu' = h^{ij}\mu'_i\mu'_j + \frac{x'}{2}\frac{\partial h^{ij}}{\partial x'}\mu'_i\mu'_j \end{aligned} \qquad \begin{aligned} &\frac{d}{ds'}\mu'_k = -\mu'_k\nu' - \frac{1}{2}\frac{\partial h^{ij}}{\partial (y')^k}\mu'_i\mu'_j \\ &\frac{d}{ds'}\kappa = 1 - \kappa\nu', \end{aligned} \tag{9.12}$$

where $ds' := x'ds$. This flow is defined for all 'time', but the vector field vanishes where $\rho = 0, \mu = 0$, so we need to desingularize about this set (i.e. blow it up). Let us introduce the coordinates $M_i = \mu'_i/\rho$, $K = (\kappa - 1)/\rho$, and consider the flow just on the energy surface $\{(\nu')^2 + h^{ij}\mu'_i\mu'_j = 1\}$. (Introduction of $M$ is equivalent to working on the bundle ${}^{\mathrm{sc}}T^*(\overline{M}) \times {}^{\mathrm{sc}}T^*(\overline{M})$ lifted to $\overline{M}_b^2$ and then blown up at $\{\rho = 0, \mu' = 0\}$). In terms of these quantities, we get, after dropping the primes on $\nu$ and $y$ and changing to a new parameter given by $ds'' = \rho\, ds'$

$$\begin{aligned} &\frac{d}{ds''}\rho = -\nu \\ &\frac{d}{ds''}y^i = h^{ij}M_j \\ &\frac{d}{ds''}\nu = \rho h^{ij}M_iM_j + \frac{\rho^2 x''}{2}\frac{\partial h^{ij}}{\partial x}M_iM_j \end{aligned} \qquad \begin{aligned} &\frac{d}{ds''}M_k = -\frac{1}{2}\frac{\partial h^{ij}}{\partial y^k}M_iM_j \\ &\frac{d}{ds''}K = \frac{h^{ij}M_iM_j}{1+\nu}. \end{aligned} \tag{9.13}$$

This remains nonsingular as long as $|M|$ remains bounded and $1 + \nu \neq 0$. For the remainder of this paragraph, let us consider what happens at the face $\mathrm{bf} = \{x'' = 0\}$, to which (9.13) is tangent. At bf, the flow is identical to the flow for the perfectly conic metric $dr^2 + r^2h(0)$. Since $d_{s''}(h^{ij}M_iM_j) = 0$ and initially, $-1 \leq |M|_h \leq 1$, $M$ certainly remains bounded. As for $\nu$, we have $\nu' \geq 0$, and under the flow (9.13), $d_{s''}\rho = -\nu$. Thus if $\rho$ becomes smaller than 1 along a trajectory, then at some point $\nu \geq 0$ and therefore $\nu > 0$ thereafter along the trajectory. Hence at $\rho = 0$, $\nu \geq 0$ and there is no singularity in the equation for $d_{s''}K$. Hence the flow at bf is well behaved in the region of interest, and it is not hard to show that it exits the region of interest in finite 'time' (measured by the parameter $s''$).

By the theory of ODEs depending on a smooth parameter, for sufficiently small initial values of $x''$ and for initial values given by (9.11), $|M|$ remains bounded and $1 + \nu$ strictly positive in the region of interest, and the region of interest is exited in finite 'time'. We see then that for small $x''$, the submanifold $L$ is smooth in the region of interest, and $K$, and hence also $\kappa$, is a smooth function on $L$.

Next we claim that in $\Omega(\eta, r_0) \cap \{\rho < 2\}$, for $\eta$ sufficiently small and $r_0$ sufficiently large, $(y', y'', \rho, x'')$ furnish coordinates on $L$. At $L \cap \{x'' = 0\}$, $\Phi$ is given by the

explicit conic formula (9.3) when $y'$ and $y''$ are close:

$$\Phi = \big((r')^2 + (r'')^2 - 2r'r'' \cos d_{\partial M}(y', y'')\big)^{1/2} = r'\big(1 + \rho^2 + 2\rho \cos d_{\partial M}(y', y'')\big)^{1/2}.$$

This is smooth away from

$$\rho = 1, \quad y' = y''. \tag{9.14}$$

Therefore the graph of $d\Phi$, restricted to $x'' = 0$, has $(y', y'', \rho)$ as smooth coordinates on any open set excluding (9.14), and hence, their differentials are linearly independent on $L \cap \{x'' = 0\}$. Also, the differential of $x''$ is nonzero when restricted to $L$ at $x'' = 0$. This follows from the fact that the differential is zero at the initial hypersurface (9.11), and that the Lie derivative of $dx''$ with respect to (9.13) vanishes. Therefore, $L$ is *projectable* — in other words, $dy', dy'', d\rho, dx''$ have linearly independent differentials when restricted to $L$ at bf, on any open set excluding (9.14), and therefore form coordinates on $L$ — *uniformly up to the corner* $\{x'' = \rho = 0\}$. By continuity, this remains true on a neighbourhood of $L \cap \{x'' = 0\}$, which covers an $\Omega(\eta, r_0)$ for sufficiently small $\eta$ and large $r_0$. Therefore, $\kappa$ is a smooth function of $(y', y'', x'', \rho)$ on $\Omega(\eta, r_0) \setminus U$ for sufficiently small $\eta$ and large $r_0$. Hence for $r_0$ sufficiently large and $\eta$ sufficiently small, there is exactly one geodesic between any two points $(r', y''), (r'', y'')$ with $r', r'' > r_0$ and $d_{\partial M}(y', y'') < \eta$ that lies wholly inside $\Omega(5\eta, r_0/2)$. Lemma 9.1 shows that every globally minimizing geodesic lies in this region. Therefore, all geodesics lying wholly within $\Omega(5\eta, r_0/2)$ are globally minimizing geodesics. This proves that $\Phi = r'\kappa$ is the genuine distance function on $M$. This completes the proof of the proposition. □

We are now able to approximate the asymptotically conic metric $d_M$ rather precisely by the perfectly conic analogue $d_{\mathrm{conic}} = d_{\mathrm{conic},h}$.

**Proposition 9.4.** *We have*

$$d_M(z', z'') = d_{\mathrm{conic}}(z', z'') + e(z', z''), \tag{9.15}$$

*where $d_{\mathrm{conic}}$ is the distance function for the perfectly conic metric $dr^2 + r^2 h_{ij}(0) dy^i dy^j$ (which is then given by (9.3)), and the error $e(z', z'')$ is $C^\infty$ on $\Omega(\eta, r_0) \setminus U$.*

*Proof.* We claim that $\kappa$ and $\kappa_{\mathrm{conic}}$ agree at each boundary hypersurface of $\overline{M}_b^2$. In fact, at lb $= \{\rho = 0\}$ they are both equal to 1, while at bf, they solve the same ODE with the same initial conditions. Equation (9.15) follows immediately. □

We can now prove Lemma 6.1. We first prove the claim for perfectly conic metrics outside of $U$.

**Lemma 9.5.** *Let $U$ be the region $U := \{(z', z'') \in M : d_M(z', z'') < (\langle z' \rangle + \langle z'' \rangle)/100\}$. Then we have*

$$|\nabla_{z'}^{m'} \nabla_{z''}^{m''} \nabla_{z', z''} d_{\mathrm{conic}}(z', z'')| \leq C_{m', m''} \langle z' \rangle^{-m'} \langle z'' \rangle^{-m''}$$

*for all $m', m'' \geq 0$ and $z', z'' \in \Omega(\eta, r_0) \backslash U$.*

*Proof.* We of course use (9.3). Without loss of generality we may take $\langle z' \rangle \geq \langle z'' \rangle$; since we are outside $U$, we thus have $d_{\mathrm{conic}}$ comparable to $\langle z' \rangle$. We begin by computing derivatives of $d^2_{\mathrm{conic}}$. Observe that $\cos d_{\partial M}(y', y'')$ is a smooth function of $d_{\partial M}(y', y'')^2$. This function is in turn smooth for $y', y'' \in B_{\partial M}(y_0, \eta)$ for $\eta$ sufficiently small (see e.g. [12]), so one can then verify the bounds

$$|\nabla_{z'}^{m'} \nabla_{z''}^{m''} d^2_{\mathrm{conic}}(z', z'')| \leq C_{m', m''} \langle z' \rangle^{1-m'} \langle z'' \rangle^{1-m''}$$

when $m' \geq 0$ and $m'' \geq 1$ (because the $(r')^2$ term vanishes); for pure $z'$ derivatives we have the slightly different formula

$$|\nabla_{z'}^{m'} d_{\mathrm{conic}}^2(z', z'')| \leq C_{m'} \langle z' \rangle^{2-m'}.$$

Expanding the latter inequality using the Leibnitz rule, one can then prove

$$|\nabla_{z'}^{m'} d_{\mathrm{conic}}(z', z'')| \leq C_{m'} \langle z' \rangle^{1-m'}$$

inductively for all $m' \geq 0$, and then by expanding the former inequality using the Leibnitz rule one can also prove

$$|\nabla_{z'}^{m'} \nabla_{z''}^{m''} d_{\mathrm{conic}}(z', z'')| \leq C_{m',m''} \langle z' \rangle^{-m'} \langle z'' \rangle^{1-m''}$$

inductively for all $m' \geq 0$ and $m'' \geq 1$. The claim follows. □

Next, we verify symbol estimates on a compact portion of the manifold.

**Lemma 9.6.** *Let $z_0 \in M$, and let $\eta > 0$ be sufficiently small depending on $z_0$. Then we have*

$$|\nabla_{z',z''}^{m} d_M(z', z'')| \leq C_{z_0,\eta,m} d_M(z', z'')^{1-m}$$

*for all $m \geq 0$ and $z', z'' \in B(z_0, \eta)$.*

*Proof.* We work in normal co-ordinates $z^j$ around $z'$, so $z^j(z') = 0$ and $\sum_j |z^j(z'')|$ is comparable to $d_M(z', z'')$. Write $\tau := d_M(z', z'')/\eta$, and rescale the co-ordinates by $\tau$ (cf. Lemma 6.2), giving rise to a new manifold $M_\tau$ with metric $g_\tau^{jk}(z) = g^{jk}(\tau z)$. In the ball $\{\sum_j |z^j| < C\eta\}$, these metrics $g_\tau$ vary smoothly as $\tau$ varies over the compact set $[0, 1]$, and the distance functions $d_{M_\tau}(z', z'')$ also vary smoothly in the region where $\sum_j |z^j(z')| < C\eta$, $\sum_j |z^j(z'')| < C\eta$, and $\sum_j |z^j(z') - z^j(z'')| > c\eta$. In particular we have estimates of the form

$$|\nabla_{z',z''}^{m} d_{M_\tau}(z', z'')| \leq C_{z_0,\eta,m}$$

uniformly in $\tau$ in this region. The claim then follows by undoing the scaling. □

Now we combine these two estimates to obtain Lemma 6.1.

*Proof of Lemma 6.1.* We let $U$ be the region $\{(z', z'') \in \Omega(\eta, r_0) : d_M(z', z'') < c(r' + r'')\}$ for some small $c > 0$, together with the boundary of this region on bf. Outside of $U$, we have $d_M(z', z'')$ comparable to $r' + r''$, and the symbol bound on $d_M$ follows from (9.15), since the error $e$ being smooth on the compactified manifold automatically obeys symbol estimates (indeed it obeys estimates which are one order of $r' + r''$ better than required), while the symbol estimates $d_{\mathrm{conic}}$ follow from Lemma 9.5. Inside the region $U$, we can then rescale as in Lemma 6.2 to rescale $z'$ and $z''$ to a fixed compact set, and then apply Lemma 9.6, if we choose $c$ sufficiently small. Note that the claim concerning the nature of the singularity of $d_M$ at the diagonal $z' = z''$ just follows from the fact that $d_M^2$ is smooth and vanishes to second order at the diagonal (see e.g. [12]). □

It remains to prove Proposition 6.3. We first show this in the case of perfectly conic manifolds (in which the non-angular error terms do not appear).

**Lemma 9.7.** *For the perfectly conic distance $d_{\mathrm{conic}}(z', z'')$, we have the estimate*[18]

$$\frac{d^2}{dt^2} d_{\mathrm{conic}}(z'(t), z''(t)) \geq -C\Big(\frac{|v'_\perp|^2}{r'} + \frac{|v''_\perp|^2}{r''}\Big), \tag{9.16}$$

*for $d_{\partial M}(y', y'')$ less than the injectivity radius of $(\partial M, h(0))$, where $z'(t)$ moves along a geodesic (for the conic metric) with initial condition $(z', v')$, and $z''(t)$ moves along a geodesic with initial condition $(z'', v'')$. Here $v'_\perp$ resp. $v''_\perp$ denotes the component of $v'$ resp. $v''$ perpendicular to both $\partial_r$ and $\frac{d}{ds}\gamma$, the tangent vector to the geodesic $\gamma$ from $z'$ to $z''$.*

Note that, in the notation of Proposition 6.3, $|v'_\perp| \leq \left|v'_{\mathrm{ang}}\right|$, $|v''_\perp| \leq \left|v''_{\mathrm{ang}}\right|$, hence this result implies Proposition 6.3 in the perfectly conic case.

*Proof.* Let us decompose the tangent vectors $v' = v'_\perp + v'_{\mathrm{par}}$, $v' = v''_\perp + v''_{\mathrm{par}}$, where $v'_{\mathrm{par}}$ is in the span[19] of $\gamma'$ and $\partial_r$, and $v'_\perp$ is perpendicular to the span of $\gamma'$ and $\partial_r$, and similarly for $v''$. Clearly the left-hand side of (9.16) is a bilinear form in these velocity variables.

First suppose that $v'_\perp = v''_\perp = 0$. The geodesic $\gamma(s)$ lies in a flat plane which is totally geodesic. Hence, in this case all geodesics $\gamma(t)$ from $z'(t)$ to $z''(t)$ lie in a flat plane, and the truth of (9.16) follows from its truth in Euclidean $\mathbf{R}^2$, which was shown in Section 2.

Next suppose that $v'_{\mathrm{par}} = v''_{\mathrm{par}} = 0$. In this case, we use the formula (9.3) and compute explicitly, exploiting the fact that in this case

$$\frac{d}{dt} d_M(z', z'')|_{t=0} = \frac{d}{dt} r'(t)|_{t=0} = \frac{d}{dt} r''(t)|_{t=0} = \frac{d}{dt} d_{\partial M}(y', y'')|_{t=0} = 0.$$

For a perfectly conic metric we have from (9.2) that

$$\frac{d^2}{dt^2}(r')^2(t) = 2|\mu'|^2, \quad \frac{d^2}{dt^2} r'(t) = |\mu'|^2/r',$$

hence $d^2/dt^2 d_{\mathrm{conic}}(z', z'')$ is given by

$$\frac{1}{d_{\mathrm{conic}}(z', z'')}\Bigg(\frac{d^2}{dt^2}((r')^2) + \frac{d^2}{dt^2}((r'')^2) - 2\big((\frac{d^2}{dt^2} r')r'' + (\frac{d^2}{dt^2} r')r''\big) \cos d_{\partial M}(y', y'') + 2r'r''\frac{d^2}{dt^2}\cos d_{\partial M}(y', y'')\Bigg) \tag{9.17}$$

Note that $(\cos d_{\partial M}(y', y'')) = (1 - d_{\partial M}(y', y'')^2/2 + O(d_{\partial M}^4))$ is a smooth function of $(y', y'')$. A computation in normal coordinates around (say) $y'$ shows that

$$\frac{d^2}{dt^2}(\cos d_{\partial M}(y', y'')) \geq -C\Big(\frac{|\mu'|}{r'} + \frac{|\mu''|}{r''}\Big)^2 d_{\partial M}(y', y'').$$

[18] We caution the reader that there are *two* 'time' variables in play here; the variable $t$ used to parameterize the moving particles $z'(t)$ and $z''(t)$, and also the implicit variable $s$ used to parameterize the geodesic connecting $z'(t)$ and $z''(t)$.

[19] If $z'$ and $z''$ lie on the same radial arc, then $\gamma'$ and $\partial_r$ are parallel, but the argument still works (e.g. by adjoining an arbitrary vector to the span of $\partial_r$ to define the $v'_{\mathrm{par}}$ component) in this case.

Since $d_{\partial M}(y', y'')$ is bounded above, we can estimate (9.17) from below by

$$\frac{1}{d_{\text{conic}}(z', z'')}\left(2(|\mu'|^2+|\mu''|^2)-2\big(\frac{r''|\mu'|^2}{r'}+\frac{r'|\mu''|^2}{r''}\big)-Cr'r''\big(\frac{|\mu'|}{r'}+\frac{|\mu''|}{r''}\big)^2\right) \tag{9.18}$$

We rewrite this as

$$\frac{1}{d_{\text{conic}}(z', z'')}\left(2\big(\frac{(r'-r'')|\mu'|^2}{r'}+\frac{(r''-r')|\mu''|^2}{r''}\big)-C\big(\frac{r''|\mu'|^2}{r'}+\frac{r'|\mu''|^2}{r''}\big)\right) \tag{9.19}$$

Since $|r'-r''| < d_{\text{conic}}(z', z'')$, the first term satisfies the estimate (9.16). For the second term, if $r' \le 2r'' \le 4r'$ then it automatically satisfies the estimate (9.16), while if the ratio of $r'$ and $r''$ is large then $d_{\text{conic}}(z', z'')$ is comparable to the *larger* of $r'$ and $r''$, showing that in this case also the second term satisfies the estimate (9.16).

Finally consider the case when one vector, say $v'$, is angular, $v'_{\text{par}} = 0$, and the other is parallel, $v''_{\perp} = 0$. In this case, we write

$$\frac{d^2}{dt^2} d_{\text{conic}} = \nabla^2_{(v',v'')} d_{\text{conic}} = \nabla^2_{(v',0)} d_{\text{conic}} + 2\nabla_{(v',0)}\nabla_{(0,v'')} d_{\text{conic}} + \nabla^2_{(0,v'')} d_{\text{conic}}.$$

and note that the geodesic $\gamma_t$ from $z'(0)$ to $z''(t)$ lies in the flat plane containing $\gamma$ for all $t$. Hence $v' = v'_{\perp}$ is perpendicular to $\gamma_t$ for all $t$, and we obtain

$$\nabla_{v'} d(z', z''(t)) = 0 \quad \text{for all } t.$$

Differentiating with respect to $t$ then shows that the cross term in (9) vanishes. The estimate (9.16) then follows, since the terms $\nabla^2_{(v',0)} d_{\text{conic}}$ and $\nabla^2_{(0,v'')} d_{\text{conic}}$ have already been treated. □

**Lemma 9.8.** *Suppose that* $((r', y'), (r'', y'')$ *are in a small neighbourhood* $U$ *of the diagonal in* $\overline{M}_b^2$, *say in the set* (9.8) *for* $\epsilon > 0$ *sufficiently small. Then the estimate* (6.1) *holds.*

*Proof.* We begin by scaling the problem so that the initial point $z'(0) = (r'(0), y'(0))$ is at $r = 1$. Let $\tau^{-1} = r'(0)$. We scale by factor $\tau$; that is, we consider the metric $g_\tau = dr^2 + r^2 h(\tau/r, y)$ and replace the geodesic $(r(t), y(t))$ by $(\tau r(t), y(t))$ which is a (reparametrized) geodesic with respect to $g_\tau$. An equivalent formulation of the lemma is to prove that

$$\begin{aligned}\frac{d^2}{dt^2} d_{\text{conic}}(z'(t), z''(t)) \ge -C\Big(|v'_{\perp}|^2 + |v''_{\perp}|^2\Big)\\ -C\tau\Big(|v'|^2 + |v''|^2\Big)\end{aligned}$$

for all geodesics joining points $(r', y')$, $(r'', y'')$ satisfying $r' = 1$, $|r'' - 1| < \epsilon$, $d_{\partial M}(y', y'') < \eta$.

To do this, we return to the second variation formula, and use the fact that

$$\frac{d^2}{dt^2} d_M(z'(t), z''(t)) \ge -\int_0^{d(z'(0),z''(0))} \langle R(T,J)T, J\rangle ds$$

where $J$ is the Jacobi field corresponding to the family of unit speed geodesics $\gamma_t(s)$ between $z'(t)$ and $z''(t)$ and $T = d\gamma/ds$. Let the boundary value of $J$ at $z'$ be decomposed as above in to $J = J'_{\perp} + J'_{\text{par}}$, and define $J''_{\perp}$ and $J''_{\text{par}}$ similarly. Let $J_1$ be the Jacobi field with boundary values $J'_{\perp}$, $J''_{\perp}$ and let $J_2$ be the Jacobi field with boundary values $J'_{\text{par}}$, $J''_{\text{par}}$. For the perfectly conic metric at $\tau = 0$, $J_2$ is in

the plane spanned by $\frac{d}{ds}\gamma$ and $\partial_r$. Hence in the general case $J_2(\tau)$ is in the plane spanned by $\frac{d}{ds}\gamma$ and $\partial_r$ up to $O(\tau)$. This means that for any $V$,

$$\langle R(T, J_2)T, V\rangle \text{ and } \langle R(T, V)T, J_2\rangle = O(\tau|J_2||V|).$$

Hence, using the fact that $d(z'(0), z''(0))$ is bounded on our rescaled region of interest,

$$\frac{d^2}{dt^2} d(z'(t), z''(t)) \geq -C(\|J_1\|_\infty^2 + \tau^2\|J_2\|_\infty^2). \tag{9.20}$$

For $\tau \in [0, 1]$, we can estimate the length of a Jacobi field corresponding to $v = (v', v'')$ by

$$\|J\|_\infty^2 \leq C(|v'|^2 + |v''|^2) \tag{9.21}$$

with a uniform constant $C$ using a comparison estimate as in the proof of Theorem 4.5.1 of [12]. Applying (9.21) separately to $J_1$ and $J_2$, we find that (9.20) implies the desired estimate. □

We can finally give the proof of Proposition 6.3, which is the final geometric lemma required for Theorem 1.5.

*Proof of Proposition 6.3.* In a conic neighbourhood $U$ of the diagonal this is precisely the content of Lemma 9.8. Outside $U$, we use Proposition 9.4 and split $d_M(z', z'')$ into the conic distance function $d_{\text{conic}}(z', z'')$ plus a symbolic error $e$. The smoothness of $e$ on $\Omega(\eta, r_0)\backslash U$ gives an error term of the form

$$\frac{|v'|^2}{(r')^2} + \frac{|v''|^2}{(r'')^2}. \tag{9.22}$$

As for the conic metric, we need to compare second derivatives of the distance function along geodesics for $g$, to second derivatives of the distance function along geodesics for $g_{\text{conic}}$. The geodesics differ by $O(t^2/r^2)$ as one can see from the geodesic equation

$$\frac{d^2}{dt^2} z^i = \Gamma^i_{jk} \frac{d}{dt} z^j \frac{d}{dt} z^k$$

since Christoffel symbols for $g$ differ from Christoffel symbols for $g_{\text{conic}}$ by $O(r^{-2})$ (with respect to a unit length frame). Hence this discrepancy also gives rise to an error of the form (9.22). Finally the conic calculation gives rise to error terms of the form (9.16). This completes the proof. □

## 10. Appendix II: Proof of local smoothing estimates

In this section we prove the local smoothing estimates (1.11) – (1.13) and Lemma 4.3. The proof of these types of estimates in manifolds usually proceeds by the positive commutator method using pseudo-differential operators which are adapted to the geometry of geodesic flow. We shall continue to follow this method, however instead of using standard pseudo-differential operators, we shall use the calculus of *scattering pseudo-differential operators* on $\overline{M}$, described by Melrose [16]. This calculus is based on the quantization of certain types of *scattering symbol classes* $S^{m,l}_{1,\rho}(\overline{M})$, which differ from standard symbols in a number of ways (in particular, near $\partial M$, every derivative in $z$ of the symbol gains a power of $r = 1/x$), and are defined as follows.

**Definition 10.1.** Let $m, l$ be real numbers and $0 \leq \rho < 1$. A smooth functions $a : {}^{\mathrm{sc}}T^*\overline{M} \to \mathbf{C}$ is said to be in the *scattering symbol class* $S^{m,l}_{1,\rho}(\overline{M})$ provided that one has the usual Kohn-Nirenberg symbol estimates

$$|\nabla_z^\alpha \nabla_\zeta^\delta a(z,\zeta)| \leq C_{\alpha,\delta,K}(1+|\zeta|)^{m-|\delta|}$$

for $z$ ranging in any compact subset $K$ of $M$, and all $\alpha, \beta \geq 0$, as well as the scattering region estimates

$$\left|\partial_x^\alpha \nabla_y^\beta \partial_\nu^\gamma \nabla_\mu^\delta a(x,y,\nu,\mu)\right| \leq C_{\alpha,\beta,\gamma,\delta}\,(1+|\nu|+|\mu|)^{m-\gamma-|\delta|} x^{l-\alpha-\rho(\gamma+|\delta|)}$$

whenever we are in the scattering region $(0,\epsilon_0)\times\partial M$, where the dual variables $\nu$, $\mu$ are as in the previous appendix. Equivalently, using radial and angular derivatives instead of $x$ and $y$ derivatives, we have

$$\left|\partial_r^\alpha \not\nabla^\beta \partial_\nu^\gamma \nabla_\mu^\delta a(1/r,y,\nu,\mu)\right| \leq C_{\alpha,\beta,\gamma,\delta}\,(1+|\nu|+|\mu|)^{m-\gamma-|\delta|} r^{-l-\alpha-|\beta|+\rho(\gamma+|\delta|)}.$$

This defines seminorms $\|\cdot\|_{\alpha,\beta,\gamma,\delta}$ on $S^{m,l}_{1,\rho}(\overline{M})$ in the usual way. Every symbol $a$ in $S^{m,l}_{1,\rho}(\overline{M})$ can be quantized to give a *scattering pseudo-differential operator $Op(a)$* in the class $\Psi^{m,l;\rho}_{\mathrm{sc}}(M)$; the exact means of quantization is not particularly important for our purposes, but we can for instance use the Kohn-Nirenberg quantization (on any $n$-dimensional asymptotically conic manifold)

$$\mathrm{Op}(a)u(z') := (2\pi)^{-n}\int \chi(z',z'')e^{i\langle -\exp_{z'}^{-1}(z''),\zeta\rangle} a(z',\zeta)u(z'')\,dz''\,d\zeta,$$

to define the kernel of $\mathrm{Op}(a)$ near $\partial M$; here, $\chi$ is a cutoff near the diagonal, chosen so that the inverse of the exponential map is well defined on $\operatorname{supp}\chi$.

*Example* 10.2. If $m, l \in \mathbf{R}$, then an operator of the form $(1+H)^{m/2}\langle z\rangle^{-l}$ or $\langle z\rangle^{-l}(1+H)^{m/2}$ lies in $\Psi^{m,l;\rho}_{\mathrm{sc}}$ for any $0 \leq \rho < 1$. Heuristically, the quantization of $a(x,y,\mu,\nu)$ can be thought of as $a(1/r, y, \frac{1}{i}\partial_r, \frac{1}{i}\not\nabla)$.

*Remark* 10.3. Scattering symbols $S^{m,l}_{1,\rho}(\overline{M})$ differ from their more standard counterparts $S^{m,0}_{0,0}(\overline{M})$ in that there is a decay of $r^{-l}$ in the symbol, that differentiation in the spatial directions gains a power of $r$, and that differentiation in the frequency directions loses a power of $r^\rho$. The case $\rho = 0$ is the most classical case and is used to prove (1.11) – (1.13); however the proof of Lemma 4.3 requires the use of more exotic symbols with $0 < \rho < 1$, so that differentiating in the frequency variables $\nu, \mu$ costs us a small power of $r$; this is due to a certain cutoff in $\mu$ which is necessary in our argument.

We now review the calculus for these scattering pseudo-differential operators, obtained[20] in [16]. If $A$ is an operator in $\Psi^{m,l;\rho}_{\mathrm{sc}}(\overline{M})$, then its symbol $\sigma(A) \in S^{m,l}_{1,\rho}(\overline{M})$ is well defined modulo a lower order symbol in $S^{m-1,l+1-\rho}_{1,\rho}(\overline{M})$, and its adjoint $A^*$ is also in this class with symbol

$$\sigma(A^*) = \overline{\sigma(A)} + O(S^{m-1,l+1-\rho}_{1,\rho}(\overline{M})) \tag{10.1}$$

where we use $O(S^{m,l}_{1,\rho}(\overline{M}))$ to denote an error in the class $S^{m,l}_{1,\rho}(\overline{M})$. If $A \in \Psi^{m,l;\rho}_{\mathrm{sc}}(\overline{M})$ and $B \in \Psi^{m',l';\rho}_{\mathrm{sc}}(\overline{M})$ then

$$AB \in \Psi^{m+m',l+l';\rho}_{\mathrm{sc}}(\overline{M}); \quad i[A,B] \in \Psi^{m+m'-1,l+l'+1-\rho;\rho}_{\mathrm{sc}}(\overline{M}). \tag{10.2}$$

[20]Strictly speaking, these results were only obtained in [16] for $\rho = 0$ but the case for $0 < \rho < 1$ follows from the same argument.

and in fact we have the more precise formulae

$$\begin{aligned} \sigma(AB) &= \sigma(A)\sigma(B) + O(S_{1,\rho}^{m+m'-1,l+l'+1-\rho}(\overline{M})) \\ \sigma(i[A,B]) &= \{\sigma(A),\sigma(B)\} + O(S_{1,\rho}^{m+m'-2,l+l'+2-2\rho}(\overline{M})) \end{aligned} \tag{10.3}$$

where $\{f,g\}$ is the usual Poisson bracket on the cotangent bundle $T^*\overline{M}$. Recall that the Poisson bracket $\{f,g\}$ may also be written as $X_f(g)$ where $X_f$ is the Hamilton vector field associated to $f$. This identification is crucial in the commutator arguments which follow.

We introduce the weighted Sobolev spaces $H^{m,l}(M)$ as

$$H^{m,l}(M) := \{u : \langle z\rangle^l u \in H^m(M)\}.$$

From (10.2), (10.3) and $L^2$-boundedness of $\Psi_{\rm sc}^{0,0;\rho}(\overline{M})$ it is easy to verify that operators in $\Psi_{\rm sc}^{m,l;\rho}(\overline{M})$ map $H^{m',l'}(M)$ to $H^{m'-m,l'+l}(M)$.

Now let $u(t)$ evolve via the flow (1.4). Sinced $e^{-itH}$ is unitary and commutes with $(1+H)^{s/2}$, we have the trivial estimate

$$\int_0^1 \|u(t)\|^2_{H^{s,0}(M)}\ dt = \|u(0)\|^2_{H^s(M)} \tag{10.4}$$

for any $s\in\mathbf{R}$. We now obtain a local smoothing estimate and a Morawetz estimate for this flow; note that we are now putting the ang$z$ weights into Sobolev norms rather than writing them explicitly:

**Lemma 10.4.** *For any $s \in \mathbf{R}$ and $\varepsilon > 0$, we have*

$$\int_0^1 \|u(t)\|^2_{H^{s,-1/2-\varepsilon}(M)}\ dt \le C_{s,\varepsilon}\|u(0)\|^2_{H^{s-1/2}(M)} \tag{10.5}$$

*and*

$$\int_0^1 \|\chi\nabla\!\!\!\!/\, u(t)\|^2_{H^{s-1,-1/2}(M)}\ dt \le C_s\|u(0)\|^2_{H^{s-1/2}(M)} \tag{10.6}$$

*where $\chi$ is a smooth cutoff to the scattering region $M\backslash K_0$.*

Observe that this lemma, combined with the above machinery, immediately gives (1.11) – (1.13).

*Proof.* We first observe that it suffices to prove these estimates for a single value of $s$, since the general case then follows by applying a power of $(1+H)$ (which commutes with $e^{-itH}$); note that any error terms arising from commuting $(1+H)$ with, for instance, $\chi\nabla\!\!\!\!/$, can be dealt with by (10.2) and (10.4).

We begin by proving the weaker estimate (1.10), which in this language becomes

$$\int_0^1 \|\varphi u(t)\|^2_{H^{s,0}(M)}\ dt \le C_{s,\varphi}\|u(0)\|^2_{H^{s-1/2}(M)} \tag{10.7}$$

for any compactly supported non-negative bump function $\varphi$. As before it suffices to prove this for a single value of $s$, say $s=0$. This estimate is essentially in [9], [10] and we only give a sketch here. Using the non-trapping hypothesis of $M$, we may construct a symbol $a(z,\zeta)\in S_{1,0}^{-1,0}(\overline{M})$ such that $\{\sigma(H),a\}>0$ for all $|\zeta|_g \ge 1$ (say), with the uniform bound $\{\sigma(H),a\} > c > 0$ when $z$ is in the support of $\varphi$. Note that the Poisson bracket $\{\sigma(H),a\}$ is nothing more than the derivative $\frac{d}{ds}a(z(s),\zeta(s))$ along the geodesic flow (9.2), which escapes to infinity

by hypothesis. See [9], [10] for the details of this standard construction. If we let $A = \mathrm{Op}(a) \in \Psi_{\mathrm{sc}}^{-1,0;0}(\overline{M})$ be the quantization of this operator, then it maps $H^{-1/2}(M)$ to $H^{1/2}(M)$, and thus

$$|\langle Au(t), u(t)\rangle_M| \leq C\|u(t)\|^2_{H^{-1/2}(M)} = C\|u(0)\|^2_{H^{-1/2}(M)}$$

for $t = 0, 1$. Thus by (2.1) we have

$$\int_0^1 \langle i[H,A]u(t), u(t)\rangle_M \, dt \leq C\|u(0)\|^2_{H^{-1/2}(M)}.$$

But from the positivity of $\{\sigma(H), a\}$ we may write

$$\{\sigma(H), a\} = c\varphi + |e|^2 + O(S^{-1,0}_{1,0}(\overline{M}))$$

where $e \in S^{0,0}_{1,0}(\overline{M})$ and $c > 0$ is a constant. Quantizing this using (10.3), (10.1) we obtain

$$i[H,A] = c\varphi + E^*E + O(\Psi_{\mathrm{sc}}^{-1,0;0}(\overline{M}))$$

for some $E \in S^{0,0}_{1,0}(\overline{M})$, where $O(\Psi_{\mathrm{sc}}^{m,l;\rho}(\overline{M}))$ denotes a scattering pseudo-differential operator in the class $\Psi_{\mathrm{sc}}^{m,l;\rho}(\overline{M})$. The term $E^*E$ gives a positive error which can then be discarded, while the contribution of the $O(\Psi_{\mathrm{sc}}^{-1,0;0}(\overline{M}))$ error is bounded by (10.4), and the claim follows.

Now we prove (10.6). Here the most convenient value of $s$ is $s = 1$. Let $a(x,y,\mu,\nu) := \psi^2\nu$, where $\psi(x,y)$ is a smooth cutoff to the region $|x| < \epsilon$, and $0 < \epsilon \ll \epsilon_0$ is a small number to be chosen later; compare with Example 2.4. Let $A$ be the quantization of $a$. Since $a \in S^{1,0}_{1,0}(\overline{M})$, we have $A \in \Psi_{\mathrm{sc}}^{1,0;0}(\overline{M})$, and we use (2.1) as before to obtain

$$\int_0^1 \langle i[H,A]u(t), u(t)\rangle_M \leq C\|u(0)\|^2_{H^{1/2}(M)}.$$

From (9.2) we see that

$$\{\sigma(H), a\} = \psi^2(xh^{jk}\mu_j\mu_k) + \frac{\psi^2}{2}x^2\frac{\partial h^{jk}}{\partial x}\mu_j\mu_k + \{\sigma(H), \psi^2\}\nu.$$

The second term is dominated by the first in the region $|x| < \epsilon$ if $\epsilon$ is sufficiently small, while the third term is compactly supported and in $S^{2,0}_{1,0}(\overline{M})$. We can thus write

$$\{\sigma(H), a\} = c\psi^2xh^{jk}\mu_j\mu_k + |e|^2 + \varphi^2 O(S^{2,0}_{1,0}(\overline{M})) + O(S^{1,0}_{1,0}(\overline{M}))$$

for some $e \in S^{1,0}_{1,0}(\overline{M})$ and some compactly supported function $\varphi(z)$. Quantizing this we obtain

$$i[H,A] = c(x^{1/2}\nabla\!\!\!\!/_j\psi)^*h^{jk}(x^{1/2}\nabla\!\!\!\!/_k\psi) + E^*E + \varphi O(\Psi_{\mathrm{sc}}^{2,0;0}(\overline{M}))\varphi + O(\Psi_{\mathrm{sc}}^{1,0;0}(\overline{M})).$$

The second term is again positive and can be discarded, while the third term can be controlled using (10.7) and the fourth term by (10.4). The claim (10.6) follows (the contribution of the region $|x| > \epsilon$ being controlled by (10.7)).

Finally we prove (10.5). Again we set $s = 1$, and now use the symbol $a := -\psi^2x^{2\varepsilon}\nu$ where $\psi$ is as before. Since $a \in S^{1,0}_{1,0}(\overline{M})$, the quantization $A$ is in $\Psi_{\mathrm{sc}}^{1,0;0}(\overline{M})$, so by (2.1) as before we have

$$\int_0^1 \langle i[H,A]u(t), u(t)\rangle_M \leq C\|u(0)\|^2_{H^{1/2}(M)}.$$

From (9.2) we have

$$\{\sigma(H), a\} = 2\varepsilon x^{1+2\varepsilon}\nu^2\psi^2 - x^{2\varepsilon}\{\sigma(H), \psi^2\nu\}.$$

From the proof of (10.6) we see that

$$\{\sigma(H), \psi^2\nu\} = x\psi^2\mu_j\mu_k O(S^{0,0}_{1,0}(\overline{M})) + \varphi^2 O(S^{2,0}_{1,0}(\overline{M})) + O(S^{1,0}_{1,0}(\overline{M}))$$

for some compactly supported $\varphi$, and thus

$$\begin{aligned} i[H, A] &= C\varepsilon(x^{1/2+\varepsilon}\partial_r\psi)^*(x^{1/2+\varepsilon}\partial_r\psi) \\ &\quad + \not\nabla^* x^{1/2}\psi O(\Psi^{0,0;0}_{\mathrm{sc}}(\overline{M}))\psi x^{1/2}\not\nabla + \varphi O(\Psi^{2,0;0}_{\mathrm{sc}}(\overline{M}))\varphi + O(\Psi^{1,0;0}_{\mathrm{sc}}(\overline{M})). \end{aligned}$$

The second error term is controlled by (10.6), the third by (10.7), and the fourth by (10.4). This proves (10.5) in the region $|x| < \epsilon$; the region $|x| > \epsilon$ is of course controlled by (10.7). □

To prove Lemma 4.3, it turns out not to be practicable to apply the positive commutator method directly, mainly because the $z$ integrals in Lemma 4.3 are not themselves positive. Instead, we use the positive commutator method to first estimate an auxiliary positive quantity, which we then use to control the expressions in Lemma 4.3. More specifically, we prove the following variant of (10.6) which only uses 'half' an angular derivative instead of a full angular derivative.

**Lemma 10.5** (Half-angular Morawetz estimate)**.** *Let $0 < \rho < 1$, and let $\phi : \mathbf{R} \to \mathbf{R}$ be a smooth non-decreasing function such that $\phi = 0$ on $(-\infty, 1]$ and $\phi = 1$ on $[2, +\infty)$. Let $b(x, y, \mu, \nu)$ be the $S^{1,1/2}_{1,\rho}(\overline{M})$ symbol*

$$b := \frac{1}{2}\phi(\frac{\epsilon_0}{x})\phi(|\mu|^2 + |\nu|^2)\phi(\frac{|\mu|}{|\nu|x^\rho})x^{1/2}\Big(\phi(\frac{|\nu|}{|\mu|})|\mu|^{1/2}|\nu|^{1/2} + |\mu|\Big) \tag{10.8}$$

*where $|\mu|^2 = |\mu|^2_{h(x)} := h^{jk}(x, y)\mu_j\mu_k$. Let $B \in \Psi^{1,1/2;\rho}_{\mathrm{sc}}(\overline{M})$ be a quantization of this operator. Then for any $H^{s+1/2}$ solution to (1.4) and any $s \in \mathbf{R}$, we have*

$$\int_0^1 \|Bu(t)\|^2_{H^s(M)}\,dt \leq C\|u(0)\|^2_{H^{s+1/2}(M)}. \tag{10.9}$$

*Here the constants depend on $\rho$ and $s$.*

*Remark* 10.6. Ignoring all the cutoffs, the important part of $b$ is the $|\mu|^{1/2}|\nu|^{1/2}$ term. Heuristically, $B$ can be thought of as the operator $r^{-1/2}|\not\nabla|^{1/2}|\partial_r|^{1/2}$, with the various cutoffs involving $\phi$ needed to avoid a singularity arising from any degeneracy of $\not\nabla$ or $\nabla$, localizing physical space to the scattering region $r > 1/\epsilon_0$, and frequency space to the region $|\not\nabla| \geq |\nabla|r^{-\rho} \geq r^{-\rho}$. This cutoff is not dangerous for us as in the region where $|\not\nabla| \leq |\nabla|r^{-\rho}$, one can effectively deduce Lemma 4.3 from (10.5) (this is why we require $\rho > 0$). The reader may wish to ignore the presence of the $\phi$ cutoffs for a first reading. The estimate (10.9) can then be thought of, very non-rigorously, as

$$\int_0^1 \|\chi|\nabla|^{1/2}|\not\nabla|^{1/2}u(t)\|^2_{H^{s,-1/2}(M)}\,dt \leq C\|u(0)\|^2_{H^{s+1/2}(M)}$$

for some appropriate cutoff $\chi$; compare this with (10.6). The crucial point here is that we do not lose an epsilon of decay as we would from (10.5).

*Proof.* As in the proof of Lemma 10.4 it suffices to verify this when $s = 0$. We again use the positive commutator method. Morally speaking, the commutant to use is $|\nabla|\mathrm{sgn}(\frac{1}{i}\partial_r)$; however this operator is too singular, and so we must use a modified version of this commutant.

Let $a$ denote the function

$$a(x, y, \mu, \nu) := \phi^2(\epsilon_0/x)\phi^2(\nu^2 + |\mu|^2)\bigg(\phi^2\big(\frac{|\mu|}{|\nu|x^\rho}\big)\phi^2(\frac{|\nu|}{|\mu|})(-|\mu|\mathrm{sgn}\nu) + C\nu\bigg),$$

where $C$ is a large constant to be chosen later. Notice that $a$ vanishes in the near region $K_0$. This can easily be verified to be a scattering symbol of order $S^{1,0}_{1,\rho}(\overline{M})$, because of all the cutoffs. Let $A \in \Psi^{1,0;\rho}_{\mathrm{sc}}(\overline{M})$ be the quantization of $a$. Applying (2.1) as in Lemma 10.4 we have

$$\int_0^1 \langle i[H, A]u(t), u(t)\rangle_M \leq C\|u(0)\|^2_{H^{1/2}(M)}.$$

We shall now factorize $i[H, A]$ in the form

$$i[H, A] = B^*B + \sum_{\text{finite}} C_i^* C_i + O(\Psi^{2,1+\delta;\rho}_{\mathrm{sc}}(\overline{M})) + O(\Psi^{1,0;\rho}_{\mathrm{sc}}(\overline{M})) \tag{10.10}$$

where $C_i \in \Psi^{1,1/2;\rho}_{\mathrm{sc}}(\overline{M})$ and $\delta > 0$; this will prove the claim since the error terms can be treated by (10.5), (10.4). From (10.3) it suffices to obtain a representation

$$\{\sigma(H), a\} = |b|^2 + \sum_i |c_i|^2 + O(S^{2,1+\delta}_{1,\rho}(\overline{M})) + O(S^{1,0}_{1,\rho}(\overline{M})). \tag{10.11}$$

Observe that $\{\sigma(H), a\}$ vanishes when $x > \epsilon_0$, and in the region $\epsilon_0 \geq x \geq \epsilon_0/4$ it is a symbol of order $S^{2,1-\rho}_{1,\rho}(\overline{M})$ as mentioned earlier, and hence in this region is also a symbol of order $S^{2,l}_{1,\rho}(\overline{M})$ for any $l$. Thus this factorization is easy to obtain in the near region, and we can focus on the scattering region $x < \epsilon_0/2$. In particular we can now ignore the cutoff $\phi(\epsilon_0/x)$. A similar argument allows us to work in the region where $|\mu|^2 + |\nu|^2 > 2$ (as $\{\sigma(H), a\}$ is certainly in $S^{1,0}_{1,\rho}(\overline{M})$ when $|\mu|^2 + |\nu|^2 < 4$), allowing us to ignore the $\phi(|\mu|^2 + |\nu|^2)$ cutoff. We thus need to consider

$$\{\sigma(H), \phi^2\big(\frac{|\mu|}{|\nu|x^\rho}\big)\phi^2(\frac{|\nu|}{|\mu|})(-|\mu|\mathrm{sgn}\nu) + C\nu\}$$

in the region $x < \epsilon_0/2$, $|\mu|^2 + |\nu|^2 > 2$.

Since the two $\phi^2$ terms are each 1 on the support on the derivative of the other, we may express this as

$$\begin{aligned} &\phi^2\big(\frac{|\mu|}{|\nu|x^\rho}\big)\phi^2(\frac{|\nu|}{|\mu|})\{\sigma(H), -|\mu|\mathrm{sgn}\nu\} + C\{\sigma(H), \nu\} \\ &-|\mu|\mathrm{sgn}\nu\,\{\sigma(H), \phi^2(\frac{|\nu|}{|\mu|})\} - |\mu|\mathrm{sgn}\nu\,\{\sigma(H), \phi^2\big(\frac{|\mu|}{|\nu|x^\rho}\big)\}. \end{aligned} \tag{10.12}$$

We have $\{\sigma(H), -|\mu|\mathrm{sgn}\nu\} = x\big(|\mu||\nu| + O(S^{2,1}_{1,\rho}(\overline{M}))\big)$ and $\{\sigma(H), \nu\} = x\big(|\mu|^2 + O(S^{2,1}_{1,\rho}(\overline{M}))\big)$. Hence for $C > 1$ the sum of these terms is larger than $|b|^2$, where $b$ is as in (10.8). To analyse the third term we compute

$$\{\sigma(H), \phi^2(\frac{|\nu|}{|\mu|})\} = 2x\phi\phi'(\frac{|\nu|}{|\mu|})\big(|\mu| + \frac{\nu^2}{|\mu|}\big) + O(S^{2,2}_{1,\rho}(\overline{M})).$$

This term is dominated by the second term, modulo acceptable errors, for sufficiently large $C$. As for the fourth term, we compute

(10.13)
$$|\mu|\{\sigma(H),\phi^2(\frac{|\mu|}{x^\rho|\nu|})\} = -2x|\mu|\phi\phi'(\frac{|\mu|}{x^\rho|\nu|})\Big(\frac{(1+\rho)|\mu|}{x^\rho}+\frac{|\mu|^3}{x^\rho\nu^2}+O((S^{1,1}_{1,\rho}(\overline{M}))\Big).$$

These terms are all supported where $|\mu|$ is comparable to $x^\rho\nu$. Since there is an overall factor of $|\mu|^2$ on the right hand side of (10.13), we gain a factor of $x^{2\rho}$, showing that (10.13) is a symbol of order $(2,1+\rho)$ which is an error term in (10.11).

Assume that the function $\phi$ is such that terms such as $\sqrt{\phi}$ and $\sqrt{\phi'}$, etc, are smooth; for example, we make take

$$\phi(t) = 1-\phi(3-t) \text{ and } \phi(t) = e^{-(t-1)^{-1}} \text{ for } t\in[1,1.2].$$

Then $\{\sigma(H),a\}$ can be written, modulo acceptable errors, as $|b|^2$ plus a sum of squares $\sum|c_i|^2$. This completes the proof of (10.10), which establishes the lemma. □

*Proof of Lemma 4.3.* We first prove (4.5). Observe if the derivative $\nabla_k$ was replaced by an angular derivative then this claim would follow from (1.13) or (10.6), so it suffices to show that

$$\int_0^1 \sup_{w\in W}\Big|\int_M \frac{a^j_w(z)\nabla\!\!\!/_j u(t,z)\overline{\partial_r u(t,z)}}{r}\,dg(z)\Big|dt \le C\|u(0)\|^2_{H^{1/2}(M)}$$

where $a^j_w$ satisfies similar symbol estimates to $a^{jk}_w$. Choose a $0<\rho<1$ (for instance, we may take $\rho := 1/2$). Observe that we may write

$$\int_M \frac{a^j_w(z)\nabla\!\!\!/_j u(t,z)\overline{\partial_r u(t,z)}}{r}\,dg(z) = \int A_w u(t,z)\overline{u(t,z)}\,dg(z)$$

where $A_w \in \Psi^{2,1;\rho}_{\mathrm{sc}}(\overline{M})$ has symbol

$$\sigma(A_w)(x,y,\nu,\mu) = Ca^j_w(x,y)x\mu_j\nu + O(S^{1,0}_{1,\rho})$$

for some constant $C$ (indeed we may improve the error term substantially). It is here that we require the functions $a^j_w$ to be symbols; otherwise $A_w$ would not be a pseudodifferential operator. Observe that in the region $|\mu|^2+|\nu|^2\le 4$, the main term is in $S^{1,0}_{1,\rho}$. From the support hypothesis of $a^j$ we thus have

$$\sigma(A_w)(x,y,\nu,\mu) = C\phi(\epsilon_0/x)\phi(|\mu|^2+|\nu|^2)a^j_w(x,y)x\mu_j\nu + O(S^{1,0}_{1,\rho}).$$

Recalling the operator $B\in\Psi^{1,1/2;\rho}_{\mathrm{sc}}(\overline{M})$ with symbol $b\in S^{1,1/2}_{1,\rho}(\overline{M})$ from the previous lemma, we can thus factorize

$$\sigma(A_w)(x,y,\nu,\mu) = O(S^{0,0}_{1,\rho}(\overline{M}))|b|^2$$
$$+ C\phi(\epsilon_0/x)\phi(|\mu|^2+|\nu|^2)(1-\phi(\frac{|\mu|}{|\nu|x^\rho}))a^j_w(x,y)x\mu_j\nu + O(S^{1,0}_{1,\rho}(\overline{M})).$$

The second term is supported on the region where $|\mu|\le 2|\nu|x^\rho$ and can thus be easily seen to lie in $S^{2,1+\rho}_{1,\rho}(\overline{M})$. Thus by (10.3) we have

$$A_w = B^*O(\Psi^{0,0;\rho}_{\mathrm{sc}}(\overline{M}))B + O(\Psi^{2,1+\rho;\rho}_{\mathrm{sc}}) + O(\Psi^{1,0;\rho}_{\mathrm{sc}});$$

since $\Psi_{\mathrm{sc}}^{m,l;\rho}$ maps $H^{m/2,-l/2}(M)$ to $H^{-m/2,l/2}(M)$,

$$\Big| \int A_w u(t,z)\overline{u(t,z)}\, dg(z)\Big| \le C\|Bu(t)\|_{L^2(M)} + C\|u(t)\|_{H^{1,-\frac{1+\rho}{2}}(M)} + C\|u(t)\|_{H^{\frac{1}{2}}(M)}$$

uniformly in $w$. The claim (4.5) then follows from Lemma 10.5, (10.5) and (10.4).

Now we prove (4.6). As before we have

$$\int_M \frac{a_w^j(z)\nabla\!\!\!/_j u(t,z)u(t,z)}{r}\, dg(z) = \int \tilde{A}_w u(t,z)\overline{u(t,z)} dg(z)$$

where $\tilde{A}_w \in \Psi_{\mathrm{sc}}^{1,1;\rho}(\overline{M})$ has symbol

$$\sigma(\tilde{A}_w)(x,y,\nu,\mu) = Ca_w^j(x,y)x\mu_j + O(S_{1,\rho}^{0,0}),$$

which is similar to the previous but without the factor of $\nu$. As before, we localize the main term to the region $|\mu|^2 + |\nu|^2 \ge 1$. We can then smoothly split the main term into the region where $|\nu| \ge \frac{1}{2}|\mu|$ and where $|\mu| \ge |\nu|$. In the first region, we have $(1+\sigma(H))^{1/2}$ comparable to $|\nu|$, and so by repeating the previous argument we eventually obtain

$$\tilde{A}_w = B^* O(\Psi_{\mathrm{sc}}^{-1,0;\rho}(\overline{M}))B + O(\Psi_{\mathrm{sc}}^{1,1+\rho;\rho}) + O(\Psi_{\mathrm{sc}}^{0,0;\rho}),$$

and one then argues as before. In the second region, we have $(1+\sigma(H))^{1/2}$ comparable to $|\mu|$, so we can write directly

$$\sigma(\tilde{A}_w)(x,y,\nu,\mu) = O(S_{1,\rho}^{-1,0})x|\mu|^2 + O(S_{1,\rho}^{0,0}),$$

and hence we have a decomposition

$$\tilde{A}_w = \nabla\!\!\!/ O(\Psi_{\mathrm{sc}}^{-1,0;\rho}(\overline{M}))\nabla\!\!\!/ + O(\Psi_{\mathrm{sc}}^{0,0;\rho})$$

uniformly in $w$. The claim now follows using (10.6) instead of Lemma 10.5. □

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